Representations of $\mathrm{GL}_2(\mathbb{F}_q)$

Kaniuar Bacho

Geboren am 18. April 1998 in Gronau (Westfalen)

Bachelorarbeit Mathematik

Betreuer: Dr. Johannes Anschütz

Zweitgutachter: Prof. Dr. Peter Scholze

Mathematisches Institut

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

0 Introduction

The goal of this thesis is to classify the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ (where $q = p^k$ is a prime power) over an algebraically closed field K of characteristic 0. More precisely we will compute the character table of $\mathrm{GL}_2(\mathbb{F}_q)$.
To achieve this goal we will develop the basic theory of group representations and of character theory in section 1 and 2. Our main references for these two sections are [Ser77] and [Ste11]. However, both of them will develop the theory only for $K = \mathbb{C}$, so we have to change some basic definitions and theorems, where complex conjugation is involved.
We will go by [JL01] for the classification of irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ in section 3.
At the end of the thesis, in section 4, we will take a look at the modular representations of $\mathrm{GL}_2(\mathbb{F}_q)$ in the smallest example (i.e. $q = 2$) to see the difference to the classical representation theory (i.e. K algebraically closed and $\mathrm{char}(K) = 0$). Good references for that would be [Ser77] and [Alp93]. We will see that our developed theory can not be applied to this case. New techniques will be necessary since the modular world behaves very different.

We will require a very good knowledge in linear algebra, group theory and in Galois theory. Everything what we need is contained in [Bos13].
Every vector $v \in K^n$ should be considered as a column vector.
Let $M_{m,n}(K)$ denote the set of $m \times n$ matrices with entries in K.
Let us introduce some notations:
If we consider a linear map f between two finite dimensional vector spaces $f : V \to W$ with ordered basis $B = (b_1, ..., b_n)$ of V respectively $C = (c_1, ..., c_m)$ of W, we denote the basis transformation matrix from B to C regarded f as $\mathbf{c}_{B,C}(f)$.
We also remember the well-known formula $\mathbf{c}_{C,D}(g) \cdot \mathbf{c}_{B,C}(f) = \mathbf{c}_{B,D}(g \circ f)$ and the linear isomorphism $\mathbf{c}_{B,C}(-) : \mathrm{Hom}_K(V, W) \xrightarrow{\sim} M_{m,n}(K)$, $f \mapsto \mathbf{c}_{B,C}(f)$. This also induces a group isomorphism $\mathbf{c}_{B,B}(-) : \mathrm{GL}(V) \xrightarrow{\sim} \mathrm{GL}_n(K)$, $f \mapsto \mathbf{c}_{B,B}(f)$.

Deutsche Zusammenfassung

In dieser Arbeit geht es um die Klassifizierung von irreduziblen Darstellungen der Gruppe $\mathrm{GL}_2(\mathbb{F}_q)$, wobei der Grundkörper K nach Annahme algebraisch abgeschlossen ist und von Charakteristik 0. Im ersten Teil der Arbeit entwickeln wir die allgemeine Darstellungstheorie von endlichen Gruppen und die Charaktertheorie. Dieses Wissen wenden wir auf unser Klassifikationsproblem an.
Im letzten Teil der Arbeit geht es um den modularen Fall der Darstellungen von $\mathrm{GL}_2(\mathbb{F}_q)$ (d.h. die Charakteristik des Grundkörpers teilt die Gruppenordnung). Aus Gründen, die im Kapitel 4 klar werden, beschränken wir uns nur auf den Fall $q = 2$. Hauptsächlich wird hier aufgezeigt, dass unsere bisherige Theorie nicht direkt anwendbar ist und nicht so einfach abstrahiert werden kann, sodass neue Methoden von Notwendigkeit sind, um modulare Darstellungen zu studieren, was man anhand von Beispielen sehen wird, da sich die modulare Welt anders verhält.

Danksagung
An dieser Stelle möchte ich mich herzlich bei allen bedanken, die mich bisher auf meinem Weg begleitet haben und viel Geduld gezeigt haben. Vor allem bei meinen Brüdern und meinem Betreuer, Dr. Johannes Anschütz, welche mir meine Unwissenheit auf vielen mathematischen Gebieten nicht tragisch genommen haben, sondern diese Unwissenheit eher mit ihrem wertvollen Wissen gefüllt haben.

Contents

1 Representation theory of finite groups

To get information about a certain group, we will enter the tremendous area of representation theory. The main idea is to get a connection to the very well-known theory of vector spaces and our algebraic structure (in this case the groups). The same idea leads to different representation theories: one of K-algebras, one of Lie algebras, and some other. However, representation theory is much more than just a means to study the structure of groups. It is also a fundamental tool with applications to many areas of mathematics and statistics, both pure and applied. It also offers a generalization of Fourier analysis via harmonic analysis (with applications in physics and number theory).
Throughout this thesis K should be a fixed field, where we will assume additional properties over time. Let us give the definition of the basic building block.

Definition 1.1. A *linear representation* of a group G is a K-vector space V together with a group homomorphism $\rho : G \to \mathrm{GL}(V)$.[1]

Of course $\mathrm{GL}(V)$ denotes the group of isomorphisms of V onto itself. The composition of the group G is denoted as $(s,t) \mapsto st$. We will also write frequently ρ_s instead of $\rho(s)$. When ρ is given, we say that V is a *representation space* of G or even simply a *representation* of G. Note that $\rho(g^{-1}) = \rho(g)^{-1}$. We will use this trivial observation often.

The main goal of representation theory is of course to determine all representations of a given group. But two at first glance different representations could be in some sense the same (by renaming the elements of the vector space). The following definition captures this behaviour.

Definition 1.2. Given two linear representations $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(V')$ of the same group G over the same field K, we define the *G-linear* (or *equivariant*)[2] morphisms $f : (\rho, V) \to (\rho', V')$ between linear representations as K-linear maps $f : V \to V'$, s.t. we have the following commutativity relation

$$f \circ \rho_s = \rho'_s \circ f \qquad \forall s \in G.$$

Another conventional way is to call f a *G-map*. If f is additionally a K-linear isomorphism (respectively bijective), we say the two linear representations are *similar* (or *isomorphic*). We can easily see that the above objects (ρ, V) and the G-linear morphisms form a category called $\mathrm{Rep}_K(G)$ (the composition of G-maps is the basic composition of linear maps). We will write for the hom-sets between two representations $\mathrm{Hom}_G((\rho, V), (\rho', V'))$, which is a subspace of $\mathrm{Hom}_K(V, V')$.
We will denote the full subcategory of representations with finite dimensional vector spaces V by $\mathrm{Rep}_K(G)^{\mathrm{fin}}$.

[1] This definition is the only obvious one: we want that G acts on a vector space V and respects the K-linear structure, this leads to a group homomorphism $G \to S(V) \cap \mathrm{End}(V) = \mathrm{GL}(V)$, where $S(V) := \{f : V \mapsto V \mid f \text{ is bijective}\}$.

[2] such a morphism is also called *intertwiner*.

Note that both categories have $(\rho = 0, V = 0)$ as a zero object, where ρ sends everything to the identity map of the zero vector space. Moreover the *coproduct* or *direct sum* of two objects (ρ, V) and (ρ', V') exists in both categories and is defined to be $(\rho \oplus \rho', V \oplus V')$, equipped with the natural embeddings inherited by the coproduct of $V \oplus V'$ in the category of vector spaces.[3] The linear representation is defined to be $(\rho \oplus \rho')_g(v, v') \coloneqq (\rho_g(v), \rho'_g(v'))$. It is an easy exercise to show that this is indeed a linear representation and a coproduct in the above categories.
This construction will play a crucial role in our theory. The main idea is to split representations into smaller one (so called irreducible representations) and study this simpler objects. An analog to group theory would be the study of simple groups (reasons are given by the Jordan–Hölder theorem).

Since everything in linear algebra has a corresponding matrix version (useful for computations), we will give two other definitions regarding matrices.

Definition 1.3. A *matrix representation* of a group G is a group homomorphism $\rho : G \to \mathrm{GL}_n(K)$ for some $n \in \mathbb{Z}_{\geq 0}$.

Here $\mathrm{GL}_n(K)$ denotes as usual the invertible $n \times n$ matrices with entries in K. So in the above setting, we are just associating every group element $s \in G$ to a matrix $R_s \in \mathrm{GL}_n(K)$ satisfying $R_s R_t = R_{st}$ for all $s, t \in G$. Note that $R_1 = 1 \in \mathrm{GL}_n(K)$ is the identity matrix and $\det(R_s) \neq 0$ for all $s \in G$.

Definition 1.4. Given two matrix representations $\rho : G \to \mathrm{GL}_n(K)$ and $\rho' : G \to \mathrm{GL}_{n'}(K)$ of the same group G, we define the *G-linear* morphisms $T : (\rho, n) \to (\rho', n')$ between matrix representations as $n' \times n$ matrices $T \in M_{n',n}(K)$ satisfying the commutativity relation

$$T \cdot R_s = R'_s \cdot T \qquad \forall s \in G.$$

If T is invertible, we say the two matrix representations are *similar* (or *isomorphic*).

We will call this category $\mathrm{Mat}_K(G)$. The hom-sets $\mathrm{Hom}_G((\rho, n), (\rho', n'))$ are again subspaces of $M_{n',n}(K)$ and the zero object is $(\rho = 0, n = 0)$. The *coproduct* or *direct sum* of two matrix representations (ρ, n) and (ρ', n') is given by $(\rho \oplus \rho', n + n')$ (we will omit the embeddings), with $(\rho \oplus \rho')_g \coloneqq \begin{pmatrix} \rho_g & 0 \\ 0 & \rho'_g \end{pmatrix}$.

The linear representations are in some sense more general since we will see that we can identify the finite dimensional linear representations with matrix representations. Playing around with these categories will show that there is some kind of same behaviour between these mathematical objects. In fact there is a deeper connection using the language of category theory:
Later on we will restrict ourselves to the case of linear representations, where V is finite dimensional. Assume for this part that V has dimension $n \in \mathbb{N}_0$ (G and the ground field K are arbitrary). We will say that n is the *degree of the representation* or *dimension of*

[3] In fact this is even a biproduct equipped with the natural projections.

the representation under consideration.
We want to show that $\mathrm{Rep}_K(G)^{\mathrm{fin}}$ is equivalent to $\mathrm{Mat}_K(G)$. Fix for every finite dimensional vector space V a basis B_V. Consider the following functor:

$$\begin{array}{rccc} F: & \mathrm{Rep}_K(G)^{\mathrm{fin}} & \longrightarrow & \mathrm{Mat}_K(G) \\ & (\rho, V) & \longmapsto & (\mathbf{c}_{B_V,B_V}(-)\circ\rho, \dim(V)) \\ & f:(\rho,V)\to(\rho',V') & \longmapsto & \mathbf{c}_{B_V,B_{V'}}(f) \end{array}$$

where $\mathbf{c}_{B,B}(-) : \mathrm{GL}(V) \xrightarrow{\sim} \mathrm{GL}_n(K)$, $f \mapsto \mathbf{c}_{B,B}(f)$ was the group isomorphism sending a linear map to the corresponding transformation matrix regarding the basis B (here $n = \dim(V)$). Now it is easy to compute that F is a functor which yields an equivalence of categories. Moreover the functor F is K-linear, i.e. on the hom-sets F is a K-linear map.
This equivalence will justify the frequently switch between linear representations and matrix representations.
There are several other equivalent definitions of representations of groups (via $K[G]$-modules or functor categories from G considered as a category to Vect_K etc.), but we will omit them since they are not of relevance in this thesis.

To achieve a good sense of the above definitions, we will take a look at some examples.

Example 1.5. A representation of degree 1 of a finite group G is (up to isomorphism) a homomorphism $\rho : G \to \mathrm{GL}_1(K) \cong K^\times$. Since G is finite with $n := |G|$, Lagrange's theorem gives us $\rho(s)^n = \rho(s^n) = \rho(1) = 1$, so every $\rho(s)$ is an nth root of unity. In particular, for a given finite group G and a fixed ground field K, there are at most finitely many non-isomorphic one dimensional representations. We can even go further and try to understand these objects:
Because of $\mathrm{im}(\rho)$ is a finite subgroup of $K^\times$, we know from a well-known fact, that $\mathrm{im}(\rho)$ is cyclic, say $\mathrm{im}(\rho) = \langle z\rangle$ for some z with $z^n = 1$. Moreover we have $\mathrm{ord}(z) = |\langle z\rangle| = |\mathrm{im}(\rho)|$, which divides $|G|$. So our options of defining ρ are again more limited.
Call this the *trivial representation* if $\rho(s) = 1 \in K^\times$ for all $s \in G$.

We can see that the tools of abstract algebra are very useful in handling with representations.

The above example is akin to the following object.

Definition 1.6. Let G be a group and K a field. Define the *dual group* $\widehat{G} \coloneqq \mathrm{Hom}(G, K^\times)$ to be the set of group homomorphism from G to $K^\times$. For two group homomorphisms $\chi_1, \chi_2 \in \widehat{G}$ define the multiplication pointwise by $(\chi_1 \cdot \chi_2)(t) = \chi_1(t) \cdot \chi_2(t)$. This yields to an abelian group structure of $\widehat{G}$.

There is a canonical group homomorphism $\varphi : G \to \widehat{\widehat{G}}$ defined by $\varphi(g)(\chi) = \chi(g)$.

Theorem 1.7. *Assume G is finite and $\mathrm{char}(K) \nmid |G|$. Then φ induces an isomorphism $G^{ab} \to \widehat{\widehat{G}}$.*

Proof. We will just sketch the proof, since this result is not important for this thesis.
There is a natural isomorphism $\widehat{G} \cong \widehat{G^{\mathrm{ab}}}$. It is easy to show that the above theorem is true if we show it only for abelian groups G. So without loss of generality assume G is abelian.
The next step is to show $G \cong \widehat{G}$ (which will not be canonical). To find such an isomorphism, use the fact that $\widehat{G \times H} \cong \widehat{G} \times \widehat{H}$ and that a finite abelian group can be decomposed into Cartesian products of cyclic groups (structure theorem for finite abelian groups). At one step you will use the isomorphism $\{x \in K^\times \mid x^n = 1\} \cong \mathbb{Z}/n\mathbb{Z}$, which is only true because $\mathrm{char}(K) \nmid |G|$ and K is algebraically closed (and n is a divisor of $|G|$).
So we showed $G \cong \widehat{G} \cong \widehat{\widehat{G}}$. Hence G and $\widehat{\widehat{G}}$ have the same cardinality, thus we only have to show injectivity of φ, i.e. $g \neq 1$ implies $\varphi(g) \neq 1$. For this just use again the structure theorem to construct a group homomorphism χ, s.t. $\chi(g) \neq 1$. □

Let us continue with other examples.

Example 1.8. Define $\zeta_n := e^{2\pi i/n} \in \mathbb{C}^\times$ to be a primitive nth root of unity. Now consider the representation $\rho : \mathbb{Z} \to \mathbb{C}^\times$ defined by $\rho(m) = \zeta_n^m$. Since the kernel satisfies $\ker(\rho) = n\mathbb{Z}$, we get a new representation $\rho' : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$ with $\rho'(\overline{m}) = \zeta_n^m$ by the homomorphism theorem.

Example 1.9. Suppose G acts on a set X by $\varphi : G \to S(X)$. Consider the vector space V over K with basis $\{e_x \mid x \in X\}$. Then φ induces a representation $\rho : G \to \mathrm{GL}(V)$ by letting ρ_g acting on the basis with $\rho_g(e_x) = e_{\varphi(g)(x)}$.
A special case is by setting $X = G$ and the action of G on itself is just the composition defined on G. This is called the *regular representation* of G.

Like every algebraic structure, we have again some substructures.

Definition 1.10. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. If W is a subspace of V and is *stable* under the action of ρ, i.e. $w \in W$ implies $\rho_s(w) \in W$ for all $s \in G$, then the restriction $\rho|_W$ of ρ to the image $\mathrm{GL}(W)$ is well-defined and leads to a new representation; W is said to be a *subrepresentation* of (ρ, V).

Definition 1.11. We call a representation (ρ, V) of G *irreducible* if it is non-zero (i.e. $V \neq 0$) and the only subrepresentations are 0 and V.

Example 1.12. Consider two representations $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(V')$ and a G-map $f \in \mathrm{Hom}_G((\rho, V), (\rho', V'))$ between them. Then $\ker(f)$ is a subrepresentation of (ρ, V) and $\mathrm{im}(f)$ is a subrepresentation of (ρ', V').

Example 1.13. Let G be finite here. Take for V the regular representation of G and let W be the subspace generated by $x := \sum_{s \in G} e_s$. We have $p_g(x) = x$ for all $g \in G$. Consequently W is a subrepresentation of V, isomorphic to the trivial representation.

We will also give an example of two representations that seems distinct, but are in truth the same (i.e. isomorphic as representations).

Example 1.14. Define $\varphi : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}_2(\mathbb{C})$ by

$$\varphi_{\overline{m}} = \begin{pmatrix} \cos\left(\frac{2\pi m}{n}\right) & -\sin\left(\frac{2\pi m}{n}\right) \\ \sin\left(\frac{2\pi m}{n}\right) & \cos\left(\frac{2\pi m}{n}\right) \end{pmatrix}$$

to be the matrix for rotation by $\frac{2\pi m}{n}$, and $\psi : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}_2(\mathbb{C})$ by

$$\psi_{\overline{m}} = \begin{pmatrix} e^{\frac{2\pi m i}{n}} & 0 \\ 0 & e^{-\frac{2\pi m i}{n}} \end{pmatrix}.$$

We will show that both are isomorphic. Let $A = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$ and so $A^{-1} = \dfrac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$. A short computation shows $A^{-1}\varphi_{\overline{m}}A = \psi_{\overline{m}}$. Hence they are isomorphic as representations.

As we already mentioned, we want to decompose representations into other representations.

Definition 1.15. A non-zero representation (ρ, V) is *decomposable* if it is isomorphic to the direct sum (coproduct) of two non-zero representations, i.e. $(\rho, V) \cong (\rho_1, V_1) \oplus (\rho_2, V_2)$.

In the literature one may see another definition via subrepresentations, but these are in fact equivalent. It will follow from the following

Theorem 1.16. *Let (ρ, V) be a representation.*
(i) *If there exist subrepresentations U_i of (ρ, V) with $1 \leq i \leq n$ and $V = \bigoplus_{i=1}^n U_i$ (internal direct sum of vector spaces), then there exist representations $(\rho_i, V_i) \cong (\rho|_{U_i}, U_i)$ with $(\rho, V) \cong \bigoplus_{i=1}^n (\rho_i, V_i)$.*
(ii) *Vice versa, if there exist representations (ρ_i, V_i) with $(\rho, V) \cong \bigoplus_{i=1}^n (\rho_i, V_i)$, then there exist subrepresentations U_i of (ρ, V) with $(\rho|_{U_i}, U_i) \cong (\rho_i, V_i)$ and $V = \bigoplus_{i=1}^n U_i$ (internal direct sum of vector spaces).*

Proof. (i) Just set $V_i = U_i$ and $\rho_i = \rho|_{U_i}$. The linear map $\varphi : \bigoplus_{i=1}^n (\rho|_{U_i}, U_i) \to (\rho, V)$ defined by $\varphi((u_1, ..., u_n)) = u_1 + ... + u_n$ is then an isomorphism of representations.
(ii) Consider the following G-maps $(\rho_i, V_i) \xrightarrow{\mathrm{ins}_i} \bigoplus_{i=1}^n (\rho_i, V_i) \xrightarrow{\sim} (\rho, V)$, where the first map is the embedding in the coproduct and the second map f is the assumed isomorphism. Now $U_i := f(\mathrm{ins}_i(V_i))$ is a subrepresentation of (ρ, V). It is easy to compute that they form an internal direct sum of V (use the explicit construction of the coproduct and the embeddings to see how the elements look like). $\square$

Corollary 1.17. *Every irreducible representation (ρ, V) is indecomposable.*

Proof. We will show the contraposition. Assume (ρ, V) is decomposable. With the previous lemma we get a subrepresentation unequal to 0 and V. $\square$

In the next section we will see under suitable conditions, that the converse of the corollary holds.

Now we will introduce some methods to construct new representations out of old ones. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. Define the *dual representation* $\rho^* : G \to \mathrm{GL}(V^*)$ by $\rho_g^*(f) := f \circ \rho_g^{-1}$. The well-definedness will follow from the next part.
We will introduce a more general construction regarding two representations (ρ, V) and (σ, W) of G. Define a representation $\rho\star\sigma : G \to \mathrm{GL}(\mathrm{Hom}_K(V, W))$ by setting $(\rho\star\sigma)_g(f) := \sigma_g \circ f \circ \rho_g^{-1}$. It is easy to verify that this is indeed a representation. If we set (σ, W) to be the trivial representation we get the dual representation.

Assume again we have two representation (ρ, V) and (σ, W) of G. Then we define the *tensor product of the representations* $\rho \otimes \sigma : G \to \mathrm{GL}(V \otimes W)$ by $(\rho \otimes \sigma)_g(v \otimes w) = \rho_g(v) \otimes \sigma_g(w)$. The concrete construction uses the universal property of the tensor product of vector spaces. To show that this is a group homomorphism, i.e. $(\rho \otimes \sigma)_{gh} = (\rho \otimes \sigma)_g \circ (\rho \otimes \sigma)_h$, just evaluate both sides on the pure tensors, since they form a generating set of the tensor product. The verification is left to the reader.
Our second construction is in some sense even a special case of the tensor product (in the finite dimensional case).

Lemma 1.18. *Given two finite dimensional representations (ρ, V) and (σ, W) of G. Then $(\rho^* \otimes \sigma, V^* \otimes W)$ is isomorphic to $(\rho \star \sigma, \mathrm{Hom}_K(V, W))$.*

Proof. Consider the linear map $\varphi : V^* \otimes W \to \mathrm{Hom}_K(V, W)$ by setting $\varphi(f \otimes w)(v) = f(v) \cdot w$ (again the existence follows from the universal property of the tensor product). We want to show that φ is even an isomorphism of vector spaces by constructing an inverse map:
Consider a basis $\{v_1, ..., v_m\}$ of V and $\{w_1, ..., w_n\}$ of W. Define $f_{i,j} \in \mathrm{Hom}_K(V, W)$ by setting $f_{i,j}(v_k) = \delta_{ik} \cdot w_j$ for $1 \leq i \leq m, 1 \leq j \leq n$. The set $\{f_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ forms a basis of $\mathrm{Hom}_K(V, W)$. Now define the inverse map by sending $f_{i,j}$ to $v_i^* \otimes w_j$, where v_i^* denotes the dual basis. This shows that φ is an isomorphism of vector spaces.
To show that φ is a G-map, one just has to plug in the definitions, which is left to the reader. $\square$

1.1 Maschke's theorem and Schur's lemma

We want now to prove one of the main results of representation theory of finite groups.

Proposition 1.19 (Maschke's theorem)**.** *Let G be a finite group and let $char(K) \nmid |G|$. Consider a representation $\rho : G \to GL(V)$. If (ρ, V) is not irreducible, then it is decomposable.*

Proof. Since (ρ, V) is not irreducible, there exists a subrepresentation $0 \subsetneq W \subsetneq V$ of (ρ, V). Let W' be an arbitrary complement of W in V (considered as vector spaces) and let $\mathrm{pr} : V \twoheadrightarrow W$ be the projection of V onto W with kernel W'. We form the average

$$\widetilde{\mathrm{pr}} := \frac{1}{|G|} \sum_{g \in G} \rho_g \circ \mathrm{pr} \circ \rho_g^{-1}.$$

Note that we can divide by $|G|$ because $\mathrm{char}(K) \nmid |G|$.
Since pr maps V into W and ρ_g preserves W we see that $\widetilde{\mathrm{pr}}$ maps V into W; we have $\rho_g^{-1}(w) \in W$ for $w \in W$, whence $(\mathrm{pr} \circ \rho_g^{-1})(w) = \rho_g^{-1}(w)$, so $(\rho_g \circ \mathrm{pr} \circ \rho_g^{-1})(w) = w$ and therefore $\widetilde{\mathrm{pr}}(w) = w$. Thus $\widetilde{\mathrm{pr}}$ is a projection of V onto W. Hence V is the internal direct sum of the subspaces W and $\widetilde{W} := \ker(\widetilde{\mathrm{pr}})$.
We want to show that $\widetilde{W}$ is even stable under ρ, this would prove the proposition.
We have the following chain of equations for all $s \in G$:

$$\rho_s \circ \widetilde{\mathrm{pr}} \circ \rho_s^{-1} = \frac{1}{|G|} \sum_{g \in G} \rho_s \circ \rho_g \circ \mathrm{pr} \circ \rho_g^{-1} \circ \rho_s^{-1} = \sum_{g \in G} \rho_{sg} \circ \mathrm{pr} \circ \rho_{(sg)^{-1}} = \widetilde{\mathrm{pr}}.$$

To show that $\widetilde{w} \in \widetilde{W}$ implies $\rho_g(\widetilde{w}) \in \widetilde{W}$ for all $g \in G$, we show $\widetilde{\mathrm{pr}}(\rho_g(\widetilde{w})) = 0$. This is indeed the case; with the above equation we get

$$\widetilde{\mathrm{pr}}(\rho_g(\widetilde{w})) = (\widetilde{\mathrm{pr}} \circ \rho_g)(\widetilde{w}) = (\rho_g \circ \widetilde{\mathrm{pr}})(\widetilde{w}) = 0.$$

This completes the proof. □

Corollary 1.20 (Completely reducible)**.** *In the above setting every finite dimensional representation (ρ, V) is a direct sum of irreducible representations.*

Proof. We proceed by induction on $\dim(V)$. If $\dim(V) = 0$ then we have the direct sum of the empty family of irreducible representations. Assume then $\dim(V) \geq 1$. If (ρ, V) is irreducible, we are done. If this is not the case, we can decompose (ρ, V) into two subrepresentations of smaller dimension by Maschke's theorem. Now use the induction hypothesis. □

Since some literature refer to this as Maschke's theorem, we will do it as well.

Remark 1.21. It is natural to ask if this decomposition is unique in some sense. This is indeed the case if K is algebraically closed and $\mathrm{char}(K) = 0$ as we will see as an application of character theory.

Maschke's theorem will simplify our theory immensely, since every finite dimensional representation is a finite direct sum of irreducible ones. So it is enough to study irreducible representations, where the theory of characters gives us powerful tools. But if the vector space is eventually infinite dimensional new methods are required. By allowing it to be, for instance, a Hilbert space, methods of analysis can be applied to the theory of groups. But we will restrict us in the next section to finite dimensional vector spaces to develop the character theory with tools of linear algebra.
We will give now an example, where Maschke's theorem can not be applied to (because the conditions are not satisfied).

Example 1.22. Let K be an infinite field with $\mathrm{char}(K) = 0$ and consider a non-zero subgroup $(G, +) \leq (K, +)$. Hence G is infinite itself. Now define the following representation $\varphi : G \to \mathrm{GL}(K^2)$ by setting $\varphi(g) = \left(\begin{smallmatrix} 1 & g \\ 0 & 1 \end{smallmatrix}\right)$. The only non-zero proper subrepresentation is $U = \{\left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right) \mid a \in K\}$ because the elements have to be eigenvectors of the image of φ. Hence this representation is not irreducible, but it is indecomposable.

We want to give two other examples, where G is finite, but the characteristic of the ground field divides the group order.

Example 1.23. Let $G = (\mathbb{F}_p, +)$ and K a field with $\mathrm{char}(K) = p$. Define the representation $\varphi : G \to \mathrm{GL}(K^2)$ by setting $\varphi(g) = \left(\begin{smallmatrix} 1 & g \\ 0 & 1 \end{smallmatrix}\right)$. Again the only non-zero proper subrepresentation is $U = \{\left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right) \mid a \in K\}$.

Example 1.24. Set $G = (\mathbb{F}_2, +)$ and let K be a field with $\mathrm{char}(K) = 2$. Define the representation $\varphi : G \to \mathrm{GL}(K^2)$ by setting $\varphi(1) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Since this matrix has only eigenvalues equal to 1, the subspace $U = \{\left(\begin{smallmatrix} a \\ a \end{smallmatrix}\right) \mid a \in K\}$ is the only non-zero proper subrepresentation.

Now we come to another essential theorem of representation theory. Schur's lemma will be used throughout this thesis.

Proposition 1.25 (Schur's lemma)**.** *Let the ground field K be algebraically closed. Consider two irreducible representations $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(V')$ and let $f \in \mathrm{Hom}_G((\rho, V), (\rho', V'))$. Then:*
(i) *If (ρ, V) and (ρ', V') are not isomorphic, we have $f = 0$.*
(ii) *If $(\rho, V) = (\rho', V')$, then f is a homothety (i.e. a scalar multiple of the identity).*

Proof. (i) We will show the contraposition of the first assertion. Assume $f \neq 0$. We know that the kernel and the image of G-maps are subrepresentations. Since both representations are irreducible the kernel and the image are either 0 or the whole space. Because of $f \neq 0$, the kernel is 0 and the image is the whole space, therefore f is an isomorphism.
(ii) Now we are in the case of the second assertion. Since K is algebraically closed f has an eigenvalue λ. Put $f' := f - \lambda \cdot \mathrm{id}_V \in \mathrm{Hom}_G((\rho, V), (\rho, V))$. The kernel of f' is not trivial, hence it is the whole space. We get $f' = 0$ and therefore $f = \lambda \cdot \mathrm{id}_V$. □

Corollary 1.26. *In the above setting we have:*
(i) $\dim(\mathrm{Hom}_G((\rho, V), (\rho', V'))) = 0$ *if the representations are not isomorphic.*
(ii) $\dim(\mathrm{Hom}_G((\rho, V), (\rho', V'))) = 1$ *if the representations are isomorphic.*

Proof. (i) This follows immediately from Schur's lemma.
(ii) We have $\mathrm{Hom}_G((\rho, V), (\rho', V')) \cong \mathrm{Hom}_G((\rho, V), (\rho, V))$ as vector spaces and Schur's lemma does the rest. □

There is another surprising application of Maschke's theorem and Schur's lemma in the following

Theorem 1.27. *Let G be finite and K algebraically closed with* $\mathrm{char(K)} \nmid |G|$. *Then the number of non-isomorphic irreducible representations of G is finite.*

Proof. Let (ρ, V) be the regular representation of G described in example 1.9. Maschke's theorem gives us a decomposition into finitely many irreducible subrepresentations, say $V = \bigoplus_{i=1}^n U_i$. Let (ρ', S) be any irreducible representation of G. Take an $s \in S \setminus \{0\}$. Define a linear map $f : V \to S$ by defining f on the basis vectors $f(e_g) = \rho'_g(s)$. It is easy to verify that this is even a G-map unequal to zero, since $f(e_1) = \rho'_1(s) = s$. Therefore an index i exists with $f|_{U_i} \neq 0$. So we get a non-zero G-map $f|_{U_i} : (\rho|_{U_i}, U_i) \to (\rho', S)$ between irreducible representations. Schur's lemma implies that they are isomorphic. Hence every irreducible representation of G is isomorphic to one of the U_j. Thus the number is finite. $\square$

Remark 1.28. Here is another proof in terms of $K[G]$-modules:
The map $f : K[G] \to S$ is defined to be $f(a) = as$. Since this map is non-zero and S is irreducible, the map has to be surjective. Hence S is a quotient module of $K[G]$. With the Jordan-Hölder theorem for K-Algebras, every irreducible $K[G]$-module appears in a composition series of the $K[G]$-module $K[G]$. Since this is finite dimensional, we have a finite length. Hence only finitely many non-isomorphic irreducible $K[G]$-modules.

2 Character theory (in char 0)

Assume for this whole section, that our ground field K is algebraically closed and $\mathrm{char}(K) = 0$. The reason for this is, that we will use Schur's lemma frequently and divide with natural numbers that we don't know exactly, so we have to restrict us to the case $\mathrm{char}(K) = 0$. Moreover every representation is now considered to be finite dimensional. This condition is essential, since we will always consider the trace of a linear map.
The group G is also considered to be finite. This yields to a richer theory, since we will use always the finiteness of G.

We will now develop the theory of characters and will restrict us to the main results, that are required to tackle our classification problem of $\mathrm{GL}_2(\mathbb{F}_q)$.

Definition 2.1. The *character* $\chi_{(\rho,V)} : G \to K$ of a representation $\rho : G \to \mathrm{GL}(V)$ is defined by setting $\chi_{(\rho,V)}(g) := \mathrm{tr}(\rho_g)$. The character of an irreducible representation is called an *irreducible character*. Instead of $\chi_{(\rho,V)}$ we will just write χ_ρ.

The importance of this map comes from the fact that it captures all important information of the representation, as we will see in the following theorems.

Remark 2.2. The character of a matrix representation is defined similarly. Using our functor F we get $\chi_{(\rho,V)} = \chi_{F((\rho,V))}$ (we will write $\chi_\rho = \chi_{F(\rho)}$). This equality will be the reason that we can switch between linear representations and matrix representations during computing the character. We will use this fact frequently.

We want to examine some properties of the character.

Lemma 2.3. *Let (ρ, V) and (ρ', V') be two representations of G. Then:*
(i) $\chi_\rho(1) = \dim(V)$,
(ii) $\chi_\rho(tst^{-1}) = \chi_\rho(s)$ *for all* $s, t \in G$,
(iii) *If (ρ, V) and (ρ', V') are isomorphic, then* $\chi_\rho = \chi_{\rho'}$,
(iv) $\chi_{\rho\oplus\rho'} = \chi_\rho + \chi_{\rho'}$,
(v) $\chi_{\rho\otimes\rho'} = \chi_\rho \cdot \chi_{\rho'}$,
(vi) $\chi_{\rho^*}(g) = \chi_\rho(g^{-1})$.

Proof. (i) We have $\chi_\rho(1) = \mathrm{tr}(\mathrm{id}_V) = \dim(V) \cdot 1_K$, where 1_K is the multiplicative identity in K. Since $\mathrm{char}(K) = 0$ the right hand side of the equality is precisely $\dim(V)$.
(ii) This follows immediately because the trace is cyclic.
(iii) Since they are isomorphic, the matrix representations $F((\rho, V))$ and $(F(\rho', V'))$ are isomorphic as well. And it is easy to see that isomorphic matrix representations have the same character, since they are conjugated by an invertible matrix and the rest follows by part (ii).
(iv) We have the equalities $\chi_{\rho\oplus\rho'} = \chi_{F(\rho\oplus\rho')} \overset{\text{(iii)}}{=} \chi_{F(\rho)\oplus F(\rho')} = \chi_{F(\rho)} + \chi_{F(\rho')} = \chi_\rho + \chi_{\rho'}$, where the second equality follows because $F(\rho \oplus \rho') \cong F(\rho) \oplus F(\rho')$, since F is an equivalence of categories.
(v) We consider an ordered basis $B = (b_1, ..., b_n)$ of V and $B' = (b'_1, ..., b'_m)$ of V', then we

write each linear map ρ_g and ρ'_g for all $g \in G$ as a matrix regarding the above basis. If we now take the lexicographic ordered basis $(b_1 \otimes b'_1, b_1 \otimes b'_2, ..., b_1 \otimes b'_m, b_2 \otimes b'_1, ..., b_n \otimes b'_{m-1}, b_n \otimes b'_m)$ of the vector space $V \otimes V'$ and write the map $(\rho \otimes \rho')_g$ regarding this basis, we get the Kronecker product of the first to matrices. Hence the claim.
(vi) The proceed is similar to that of (v). Take a basis B of V and write ρ_g as a transformation matrix regarding this basis. Consider the dual basis of B and write the matrix entries of ρ_g^* in terms of the first transformation matrix. The claim will follow. $\square$

These are some facts that will be used often to reduce computations to a minimum. One could ask if the converse of statement (iii) in lemma 2.3 is also true. This is in fact the case. But we have to develop a little bit more theory to prove this.

We will now try to understand the fixed points of a representation. Consider a representation (ρ, V) and define the *fixed subspace*

$$V^G := \{v \in V \mid \rho_g(v) = v \ \forall g \in G\}.$$

This is a subrepresentation of (ρ, V). We will now investigate the operator

$$P := \frac{1}{|G|} \sum_{g \in G} \rho_g \in \mathrm{End}_K(V).$$

It is easy to see that $P(v) \in V^G$ and $P^2 = P$. Hence $P : V \to V^G$ is a projection of V onto V^G. We conclude $\dim(V^G) = \mathrm{rank}(P)$ and $V = \ker(P) \oplus \mathrm{im}(P)$. Take a basis of $\ker(P)$ and one of $\mathrm{im}(P)$, the union of them is a basis of V. The trace of P regarding this basis is exactly $\dim(\mathrm{im}(P))$, therefore we get the following

Lemma 2.4. *Let (ρ, V) be a representation, then*

$$\dim(V^G) = \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}(\rho_g).$$

2.1 Class functions and Schur orthogonality relations

The idea of this section is to abstract the behaviour of characters to arbitrary maps $f : G \to K$ that satisfy property (ii) in lemma 2.3.

Definition 2.5. Let G be a group and define the *group algebra*

$$K[G] = K^G = \{f \mid f : G \to K\},$$

which is a K-vector space with the pointwise addition and obvious scalar multiplication. Moreover we will consider the following *inner product*

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) f_2(g^{-1}).$$

Remark 2.6. The inner product is in fact only a symmetric K-bilinear form. In the literature you will often find the case, where the theory is developed only for $K = \mathbb{C}$ and instead of taking the inverse in the inner product, they take the complex conjugated. But we want to develop the theory as much as abstract, so taking the inverse will help us here.

Remark 2.7. There are two ways to define the group algebra; the above definition is one of them. The other definition is to consider a vector space with basis elements $g \in G$, so the elements of $K[G]$ are linear combinations of $g \in G$ as formal sums.
Both K-vector spaces possess a K-algebra structure in a natural way. Moreover on both exist a K-bilinear form.
Most books use both ways for different applications. Our definition is more natural to define the K-bilinear form. The second definition is more natural to define a K-algebra structure and regard a representation as a $K[G]$-module.
Both definitions leads to the same object (isomorphism between K-algebras respecting the K-bilinear form), but one definition may be better than the other one if you want to visualize a certain situation.

Of interest are the following type of maps of $K[G]$.

Definition 2.8. A map $f \in K[G]$ is called a *class function* if $f(g) = f(hgh^{-1})$ for all $g, h \in G$. This means f is constant on the conjugacy classes of G. We will denote this set with $Z(K[G])$.

Remark 2.9. The K-vector space $K[G]$ is even an associative unital K-algebra, where the multiplication is the *convolution* defined by

$$(f_1 \cdot f_2)(t) = \sum_{g \in G} f_1(g) f_2(g^{-1}t)$$

One can compute that all K-algebra axioms are satisfied.
The center of this K-algebra is precisely the set of class functions $Z(K[G])$. For proofs of these statements take a look at [Ste11, p. 52-54]. Using the other definition of a group algebra is much more comfortable to prove these statements.

Assume now G is divided into its conjugacy classes $C_1, ..., C_n$ and let $Cl(G)$ be the set of conjugacy classes of G. Define the following class functions $\delta_{C_i} : G \to K$ by

$$\delta_{C_i}(g) = \begin{cases} 1 & g \in C_i \\ 0 & g \notin C_i \end{cases}$$

The set $\{\delta_{C_i} \mid 1 \leq i \leq n\}$ is then obviously a basis of $Z(K[G])$, so $\dim(Z(K[G])) = |Cl(G)|$.

We will now see the first appearance of characters and the inner product in the following

Proposition 2.10. *Let (ρ, V) and (σ, W) be two representations of G. Then $\langle \chi_\rho, \chi_\sigma \rangle = \dim(\mathrm{Hom}_G((\rho, V), (\sigma, W)))$.*

Proof. We will use all the facts that we achieved until now, especially lemmas 1.18, 2.3 and 2.4.

$$\begin{aligned}\langle \chi_\rho, \chi_\sigma \rangle = \langle \chi_\sigma, \chi_\rho \rangle &= \frac{1}{|G|}\sum_{g\in G} \chi_\sigma(g)\chi_\rho(g^{-1}) \\ &= \frac{1}{|G|}\sum_{g\in G} \chi_\sigma(g)\chi_{\rho^*}(g) \\ &= \frac{1}{|G|}\sum_{g\in G} \chi_{\rho^*\otimes\sigma}(g) \\ &= \frac{1}{|G|}\sum_{g\in G} \chi_{\rho\star\sigma}(g) \\ &= \dim(\mathrm{Hom}_K(V,W)^G) \\ &= \dim(\mathrm{Hom}_G((\rho,V),(\sigma,W))).\end{aligned}$$

□

Corollary 2.11 (Schur orthogonality relations)**.** *If* (ρ, V) *and* (σ, W) *are two irreducible representations, then:*
(i) $\langle \chi_\rho, \chi_\sigma \rangle = 0$ *if they are not isomorphic.*
(ii) $\langle \chi_\rho, \chi_\sigma \rangle = 1$ *if they are isomorphic.*

Proof. This follows immediately with the previous proposition and corollary 1.26. □

Let us now introduce some notation and state the unique decomposition of a representation.
Let $m \in \mathbb{Z}_{\geq 0}$ and (ρ, V) be a representation, then we set

$$m(\rho, V) := \bigoplus_{i=1}^{m} (\rho, V).$$

It is easy to show that we have an isomorphism of vector spaces

$$\mathrm{Hom}_G((\rho,V),(\rho_1,V_1)\oplus(\rho_2,V_2)) \cong \mathrm{Hom}_G((\rho,V),(\rho_1,V_1)) \oplus \mathrm{Hom}_G((\rho,V),(\rho_2,V_2))$$

and

$$\mathrm{Hom}_G((\rho_1,V_1)\oplus(\rho_2,V_2),(\rho,V)) \cong \mathrm{Hom}_G((\rho_1,V_1),(\rho,V)) \oplus \mathrm{Hom}_G((\rho_2,V_2),(\rho,V))$$

We know that there are only finitely many non-isomorphic irreducible representations (the trivial representation is always irreducible), as we have seen in theorem 1.27.
Take a complete set $\{(\rho_i, V_i) \mid 1 \leq i \leq n\}$ of representatives of the isomorphic classes of irreducible representations of G.
By Maschke's theorem every representation (ρ, V) is isomorphic to $\bigoplus_{i=1}^{n} m_i(\rho_i, V_i)$, where

m_i are non-negative integers. We will show that these m_i are uniquely determined. We have for all $1 \leq k \leq n$ the equalities

$$\dim(\mathrm{Hom}_G((\rho_k, V_k), (\rho, V))) = \dim(\mathrm{Hom}_G((\rho_k, V_k), \bigoplus_{i=1}^{n} m_i(\rho_i, V_i))) = m_k,$$

where the last equality follows from our above observation regarding direct sums and corollary 1.26. This means that m_i is determined by the left hand side (which is independently of the decomposition), thus the decomposition is unique up to isomorphism (i.e. replacing irreducible representations by isomorphic irreducible representations).

Remark 2.12. One could also use character theory to show this:
$\langle \chi_\rho, \chi_{\rho_k} \rangle = \langle \sum_{i=1}^{n} m_i \chi_{\rho_i}, \chi_{\rho_k} \rangle = \sum_{i=1}^{n} m_i \langle \chi_{\rho_i}, \chi_{\rho_k} \rangle = m_k$ in K. Since $\mathrm{char}(K) = 0$ we have $\mathbb{Z} \subset K$, so the natural number m_k is determined by the left hand side.

The character of a representation is very useful if we want to show that a representation is irreducible or not. Take a look at the

Proposition 2.13. *A representation (ρ, V) of G is irreducible if and only if $\langle \chi_\rho, \chi_\rho \rangle = 1$.*

Proof. The only if direction follows immediately from corollary 1.26 and proposition 2.10. For the if direction decompose (ρ, V) into irreducible representations $\bigoplus_{i=1}^{n} m_i(\rho_i, V_i)$, this leads to

$$1 = \langle \chi_\rho, \chi_\rho \rangle = \langle \sum_{i=1}^{n} m_i \chi_{\rho_i}, \sum_{i=1}^{n} m_i \chi_{\rho_i} \rangle = m_1^2 + ... + m_n^2$$

as a sum of integers in $\mathbb{Z}$. This is only the case if one of the m_i is 1 and the rest is 0. Hence our assertion. ☐

There is another important proposition to determine if two given representations are isomorphic or not.

Proposition 2.14. *Two representations (ρ, V) and (ρ', V') are isomorphic if and only if $\chi_\rho = \chi_{\rho'}$.*

Proof. We saw already the only if direction in lemma 2.3.
For the if direction decompose both representations into their irreducible representations, say $(\rho, V) \cong \bigoplus_{i=1}^{n} m_i(\rho_i, V_i)$ and $(\rho', V') \cong \bigoplus_{i=1}^{n} k_i(\rho_i, V_i)$. Then for all $1 \leq j \leq n$ we get

$$m_j = \langle \chi_\rho, \chi_{\rho_j} \rangle = \langle \chi'_\rho, \chi_{\rho_j} \rangle = k_j.$$

So the representations have the same decomposition. ☐

This proposition is the reason to say that the character of a representation encodes all information about the representation itself. Since for every representation there is a character and vice versa given a character there exists exactly one representation with that character (up to isomorphism). So the above proposition reduces the study of representations to that of their characters.

We are now ready to force the problem that the number of conjugacy classes equals the number of irreducible representations (or equivalently the number of irreducible characters).

Lemma 2.15. *Let $f \in Z(K[G])$ be a class function on G and (ρ, V) a representation of G. Define $\rho_f := \sum_{g \in G} f(g^{-1})\rho_g \in \mathrm{Hom}_K(V, V)$. Then:*
(i) $\rho_f \in \mathrm{Hom}_G(V, V)$
(ii) *If (ρ, V) is irreducible of degree n, then $\rho_f = \frac{|G|}{n}\langle f, \chi_\rho\rangle \cdot \mathrm{id}_V$*

Proof. (i) This is obvious.
(ii) With Schur's lemma we get $\rho_f = \lambda \cdot \mathrm{id}_V$. Taking the trace on both sides yields to

$$\lambda \cdot n = \mathrm{tr}(\lambda \cdot \mathrm{id}_V) = \mathrm{tr}(\rho_f) = \sum_{g \in G} f(g^{-1})\,\mathrm{tr}(\rho_g) = |G| \cdot \langle f, \chi_\rho\rangle.$$

This completes the proof. □

Lemma 2.16. *Let $\chi_1, ...\chi_n$ be all pairwise distinct irreducible characters of G. Assume there exists a class function $f \in Z(K[G])$ with $\langle f, \chi_i\rangle = 0$ for all i. Then $f = 0$.*

Proof. Let (ρ, V) be a representation of G. Consider the G-map ρ_f that we constructed in the previous lemma. If (ρ, V) is irreducible, then $p_f = \frac{|G|}{n}\langle f, \chi_\rho\rangle \cdot \mathrm{id}_V = 0$, since f is orthogonal to all irreducible characters. We will show that $\rho_f = 0$ for arbitrary representations (ρ, V):
We can decompose V into an internal direct sum of irreducible subrepresentations U_i with $1 \leq i \leq m$. Because of $\rho_f|_{U_i} = (\rho|_{U_i})_f = 0$ we get the result.
Now take for (ρ, V) the regular representation of G and evaluate ρ_f at the basis vector e_1:

$$0 = \rho_f(e_1) = \sum_{g \in G} f(g^{-1})\rho_g(e_1) = \sum_{g \in G} f(g^{-1})e_g,$$

which means $f(g^{-1}) = 0$ for all $g \in G$, i.e. $f = 0$. □

Now we are ready to achieve our goal.

Proposition 2.17. *Let $\chi_1, ...\chi_n$ be all pairwise distinct irreducible characters of G. Then they form a basis of the class functions $Z(K[G])$.*

Proof. We already know that the irreducible characters form an orthonormal system, thus they are linearly independent. We only have to show that they generate the space of class functions. Let U be the subspace generated by the irreducible characters. Consider the following linear maps

$$Z(K[G]) \xrightarrow{f_1} \mathrm{Hom}_K(Z(K[G]), K) \xrightarrow{f_2} \mathrm{Hom}_K(U, K)$$

by setting $f_1(f) = \langle -, f\rangle$ and $f_2(f) = f|_U$. It is obvious that f_2 is surjective and by lemma 2.16 $f_2 \circ f_1$ is injective. Hence f_1 is injective between two finite dimensional vector spaces of same dimension, thus f_1 is an isomorphism and therefore $f_2 \circ f_1$ is also surjective, consequently $f_2 \circ f_1$ is an isomorphism. So we get $\dim(Z(K[G])) = \dim(\mathrm{Hom}_K(U, K)) = \dim(U)$, which implies that $U = Z(K[G])$. □

This proposition proves that the number of non-isomorphic irreducible representations are precisely $\dim(Z(K[G])) = |Cl(G)|$.

Corollary 2.18. *Let f be a class function of G. Then f is the character of a representation (ρ, V) if and only if f is a linear combination of the irreducible characters with non-negative integer coefficients.*

This corollary will be useful, when we will classify the irreducible characters of $\mathrm{GL}_2(\mathbb{F}_q)$. As we saw the characters of representations encodes the information of the representation itself. We only care about irreducible characters, since all other are linear combinations with non-negative integer coefficients. Moreover every character is constant on a conjugacy class. And the number of the irreducible characters are precisely the number of conjugacy classes. So the following definition arises naturally.

Definition 2.19. Let G be a finite group with irreducible characters $\chi_1, ..., \chi_n$ and conjugacy classes $C_1, ..., C_n$. The *character table* of G is a $n \times n$ matrix X with $X_{ij} = \chi_i(C_j)$. The rows of X are indexed by the characters of G, the columns by the conjugacy classes of G and the $ij-$entry is the value of the ith-character on the jth-conjugacy class.

This character table contains all relevant information about the representations of a certain group.

Example 2.20. Let us compute the character table of the permutation group S_3. We know that S_3 has three conjugacy classes with representatives $\mathrm{id}, (12)$ and (123), where we use the cyclic notation of permutations. We have two irreducible representations of degree one. The first is the trivial representation; the second is the sgn function, that gives the parity of a permutation back to us.
Consider now the obvious group action of S_3 on $X = \{1, 2, 3\}$. Then we get a group representation (ρ, V) described in example 1.9. Let $\{v_1, v_2, v_3\}$ be the basis of V. Consider the subspace $U = \langle v_1 - v_2, v_2 - v_3 \rangle$. This is even a subrepresentation (just compute the action of $\rho_{(12)}$ and that of $\rho_{(123)}$ on it, since these two permutations generate S_3). We want to show that this subrepresentation is irreducible. We will do it by hand and without characters, since this example serves as another example in section 4.
Assume we have a subrepresentation of degree one, say $\langle av_1 + (b-a)v_2 - bv_3 \rangle$ with $a, b \in K$ not both equals zero. We will go by case distinction:
Assume $b = 0$. Then $a \neq 0$. But we have by assumption $\rho_{(123)}(av_1 - av_2) = av_2 - av_3 \in \langle av_1 - av_2 \rangle$, hence $a = 0$, contradiction.
If $b \neq 0$, then $\rho_{(12)}(av_1 + (b-a)v_2 - bv_3) = (b-a)v_1 + av_2 - bv_3 \in \langle av_1 + (b-a)v_2 - bv_3 \rangle$, hence $b = 2a$ and $a \neq 0$. But then $\rho_{(123)}(av_1 + av_2 - 2av_3) = -2av_1 + av_2 + av_3 \in \langle av_1 + av_2 - 2av_3 \rangle$, which implies $3a = 0$ and so $a = 0$, contradiction.
We showed that U is indeed an irreducible representation of degree 2. So we found all irreducible representations.
The character table looks as follows:

	id	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Table 1: Character table of S_3

2.2 Induced characters and Frobenius Reciprocity

Until now our attention was only restricted to the representations of a certain group G. It is natural to ask if there are some relations between representations of different groups. We will only scratch the surface of this theory/question. Here we just prove enough to help us classify the irreducible characters of $\mathrm{GL}_2(\mathbb{F}_q)$. We will use ad hoc methods to prove the existence of induced characters, without really revealing the nature of the induced representation.

The most obvious way to investigate this question is to compare representations of G and representations of subgroups of G. We saw that $K[G]$ plays an important role in character theory. We want to compare it with $K[H]$, where $H \leq G$ is a subgroup. The following definition will help us.

Definition 2.21. If $f : G \to K$ is a map in $K[G]$ and $H \leq G$ a subgroup, then define the *restriction* $\operatorname{Res}_H^G f : H \to K$ by setting $\operatorname{Res}_H^G f(h) = f(h)$.

Theorem 2.22. *Let $H \leq G$. Then $\operatorname{Res}_H^G : Z(K[G]) \to Z(K[H])$ is a linear map.*

Proof. The map is obviously well-defined and linear. □

Our goal is now to define a linear map going the other direction, which is at first glance not so obvious to define.[4]
If $H \leq G$ and $f : H \to K$ is a map, then define $\dot{f} : G \to K$ by

$$\dot{f}(x) = \begin{cases} f(x) & x \in H \\ 0 & x \notin H \end{cases}$$

We are going to lift class functions now via the *induction* map $\operatorname{Ind}_H^G : Z(K[H]) \to Z(K[G])$ by the formula

$$\operatorname{Ind}_H^G f(g) = \frac{1}{|H|} \sum_{x \in G} \dot{f}(x^{-1} g x).$$

The map is obviously well-defined and linear.
In the case χ is a character of H, one calls $\operatorname{Ind}_H^G \chi$ the *induced character* of χ on G. Our goal is to show that this is indeed again a character, but we will not introduce the notion

[4]It will be the character of the so called induced representation which is just defined by extension of scalars using the language of $K[G]-$modules.

of an induced representation.
We met two operators, where the first restricts to a subgroup and the second lifts to the original group. The following theorem, known as Frobenius reciprocity, asserts that the linear maps Res_H^G and Ind_H^G are adjoint.[5]
From now on we will index the inner product by the group where it should be taken in, since we are operating with two different groups.

Proposition 2.23 (Frobenius reciprocity, [Ste11, p. 99]). *Suppose $H \leq G$ and f is a class function on H and g a class function on G. Then the formula*

$$\langle \operatorname{Ind}_H^G f, g \rangle_G = \langle f, \operatorname{Res}_H^G g \rangle_H$$

holds.

Corollary 2.24. *If χ is a character of some representation (ρ, V) of H, then $\operatorname{Ind}_H^G \chi$ is a character of some representation of G.*

Proof. Since $\operatorname{Ind}_H^G \chi$ is a class function, it is a linear combination of the irreducible characters of G, say $\chi_1, ..., \chi_n$, i.e. $\operatorname{Ind}_H^G \chi = \sum_{i=1}^n a_i \cdot \chi_i$ with $a_i \in K$. Then Frobenius reciprocity yields to

$$a_k = \langle \operatorname{Ind}_H^G \chi, \chi_k \rangle_G = \langle \chi, \operatorname{Res}_H^G \chi_k \rangle_H.$$

Note that if (σ, W) is a representation of G, then $\operatorname{Res}_H^G \chi_\sigma$ is the character of $\sigma|_H$. Hence the right hand side of the equality is a non-negative integer by proposition 2.10. Now apply corollary 2.18. □

[5]With the right notation these are two adjoint functors between $\operatorname{Rep}_K(G)$ and $\operatorname{Rep}_K(H)$. Using proposition 2.10 would prove Frobenius reciprocity immediate.

3 Representations of $\mathrm{GL}_2(\mathbb{F}_q)$ (in char 0)

We arrived now to our main problem of this thesis: the classification of irreducible characters (representations) of $\mathrm{GL}_2(\mathbb{F}_q)$ over an algebraically closed field K with $\mathrm{char}(K) = 0$. To determine the character table of $\mathrm{GL}_2(\mathbb{F}_q)$, we will explore the group $\mathrm{GL}_2(\mathbb{F}_q)$ in the next section.

3.1 Structure of the group $\mathrm{GL}_2(\mathbb{F}_q)$

Let us repeat the following simple fact:

Lemma 3.1. *The order of* $\mathrm{GL}_2(\mathbb{F}_q)$ *is* $(q^2-1)(q^2-q)$.

Proof. To see this use the fact that a square matrix is invertible if and only if the columns are linear independent. So the first column has to be linear independent to itself, i.e. there are (q^2-1) possibilities (only the zero vector is not linear independent to itself). The second column should not be in the linear span of the first column, so out of q^2 possibilities we have to subtract q possibilities (the q linear combinations of the first column). All in all we get the desired order of $\mathrm{GL}_2(\mathbb{F}_q)$. □

To determine the conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$, we will first take a look at the properties of finite fields.
Remember that every finite field has an order equals a prime power. Moreover there is exactly one finite field of order $q = p^k$ (up to isomorphism), where p is a prime number and k a positive integer.
One may construct a finite field of order $q = p^k$ by considering the splitting field L of the polynomial $f = X^q - X \in \mathbb{F}_p[X]$ (L consists of the q distinct roots of f in the algebraic closure $\overline{\mathbb{F}_p}$). For a detailed description take a look at [Bos13, p. 126-129]
In the following we will consider the field extension $\mathbb{F}_q \subset \mathbb{F}_{q^2}$.

Lemma 3.2. *If* $r \in \mathbb{F}_{q^2}$ *then* $r + r^q, r^{1+q} \in \mathbb{F}_q$.

Proof. WLOG $r \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Then the minimal polynomial of r over $\mathbb{F}_q$ is of degree 2, say $\mu_r(X) = X^2 + aX + b \in \mathbb{F}_q[X]$. Let $k := \nu_p(q)$ be the p-adic valuation of q. Since we have a Galois extension of degree 2 with generator F_p^k (composition of the Frobenius homomorphism kth time with itself), the other root of $\mu_r(X)$ is r^q. Hence $a = -(r + r^q)$ and $b = r^{1+q}$. □

Since every finite subgroup of the multiplicative group of a field is cyclic, we know that $(\mathbb{F}_{q^2}^{\times}, \cdot)$ is cyclic; let ε be a generator of order (q^2-1). The roots of $f = X^{q^2-1} - 1 \in K[X]$ form also a multiplicative finite subgroup of $(K^{\times}, \cdot)$, hence we get a root ω of f with order (q^2-1). Thus every element $r \in \mathbb{F}_{q^2}^{\times}$ may be written as ε^m for some integer m (note that m is uniquely determined modulo q^2-1). Let $\bar{r} := \omega^m$ (this is well-defined since ω has also order q^2-1). The map $\bar{\cdot} : \mathbb{F}_{q^2}^{\times} \to K^{\times}$ is then an irreducible representation of degree one (the character coincide with the representation). Moreover every irreducible representation of $\mathbb{F}_{q^2}^{\times}$ is of the form $r \mapsto \bar{r}^j$ for $1 \leq j \leq q^2-1$ (since the group is abelian, there are exactly q^2-1 irreducible characters and our presented characters above are all different). Denote the above map by σ_j.

Let us introduce some notation. Let G be a finite group. For two elements $x, y \in G$ we will write $x \sim y$ if they are conjugated to each other. Let $x^G := \{gxg^{-1} \mid g \in G\}$ be the conjugacy class of x in G and let $C_G(x) := \{g \in G \mid gxg^{-1} = x\}$ be the centralizer of x. It is a well-known fact that

$$|x^G| = \frac{|G|}{|C_G(x)|}.$$

Let us go forward to the classification of conjugacy classes. Keep the following in mind:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}$$

are conjugated to each other only if $\{a, c\} = \{a', c'\}$, since their eigenvalues have to coincide.
We will now give four families of conjugacy classes where $\mathrm{GL}_2(\mathbb{F}_q)$ divides into.
The hidden idea of this whole section is to do case distinction by the minimal polynomial regarding the number of roots. For minimal polynomials of degree 2 without any root, a quadratic field extension will contain both roots, where one of the root is received by using qth time the Frobenius homomorphism (generator of the Galois group of this field extension) on the other root. This leads as a motivation for the fourth family that we will regard.

<u>First family I</u>: This is the set $I = \{s \cdot I = \left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right) \mid s \in \mathbb{F}_q^\times\}$; equals the center of $\mathrm{GL}_2(\mathbb{F}_q)$. These elements give us $q-1$ conjugacy classes of size 1.

<u>Second family U</u>: This is the set $U = \{u_s := \left(\begin{smallmatrix} s & 1 \\ 0 & s \end{smallmatrix}\right) \mid s \in \mathbb{F}_q^\times\}$. Every element gives us distinct conjugacy classes (their eigenvalues are not the same).
We want to compute the size of the conjugacy classes by computing the size of the centralizer and using the formula above.
Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{F}_q)$. It is easy to see that $g \cdot u_s = u_s \cdot g$ if and only if $g = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$. This means $|u_s^G| = \frac{|G|}{|C_G(u_s)|} = \frac{(q^2-1)(q^2-q)}{(q-1)q} = q^2-1$.
So we have $q-1$ new conjugacy classes each of size q^2-1 (they do not coincide with the first family of conjugacy classes because of obvious reasons).

<u>Third family D</u>: This is the set $D = \{d_{s,t} := \left(\begin{smallmatrix} s & 0 \\ 0 & t \end{smallmatrix}\right) \mid s, t \in \mathbb{F}_q^\times \text{ and } s \neq t\}$. Note that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdot d_{s,t} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = d_{t,s}$$

and $d_{s,t} \sim d_{s',t'}$ if and only if $d_{s',t'} = d_{s,t}$ or $d_{s',t'} = d_{t,s}$. So we get for every element a conjugacy class, where we counted each distinct conjugacy class twice because of the above conjugation of $d_{s,t} \sim d_{t,s}$. Hence we have $\frac{(q-1)(q-2)}{2}$ new conjugacy classes (different from the first and second family, since there are two eigenvalues).

We want to calculate again the size of each centralizer.
It is easy to compute that $g \cdot d_{s,t} = d_{s,t} \cdot g$ if and only if $g = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$. Hence $|d_{s,t}^G| = \frac{|G|}{|C_G(d_{s,t})|} = \frac{(q^2-1)(q^2-q)}{(q-1)^2} = q(q+1)$ is the size of each conjugacy class.

<u>Fourth family V</u>: This is the set $V = \{v_r := \left(\begin{smallmatrix} 0 & 1 \\ -r^{1+q} & r+r^q \end{smallmatrix}\right) \mid r \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q\}$. First note that this is well-defined because of lemma 3.2.
The characteristic polynomial of v_r is $p_{v_r}(X) = X^2 - (r + r^q)X + r^{1+q} \in \mathbb{F}_q[X]$, where it splits over the field $\mathbb{F}_{q^2}$ into $p_{v_r}(X) = (X - r)(X - r^q)$. So our matrix v_r has no eigenvalues in $\mathbb{F}_q$, therefore the conjugacy class is distinct to the previous three families of conjugacy classes.
Note that $v_r \sim v_{r'}$ in $\mathrm{GL}_2(\mathbb{F}_q)$ if and only if $r' = r$ or $r' = r^q$. More over $r \neq r^q$ since $r \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. So we get for every element a conjugacy class, where we counted each distinct conjugacy class twice (that of $v_r \sim v_{r^q}$). Hence we have $\frac{q^2-q}{2}$ new conjugacy classes.
To compute the size of the centralizer consider the two matrices:

$$g \cdot v_r = \begin{pmatrix} -br^{1+q} & a + b(r + r^q) \\ -dr^{1+q} & c + d(r + r^q) \end{pmatrix}$$

and

$$v_r \cdot g = \begin{pmatrix} c & d \\ -ar^{1+q} + c(r + r^q) & -br^{1+q} + d(r + r^q) \end{pmatrix}$$

A necessary condition would be $c = -br^{1+q}$ and $d = a + b(r + r^q)$. This is also sufficient under the assumption $(a, b) \neq (0, 0)$, as we will see:
The equality of the matrix is clear, but we have to show that g is indeed invertible. The determinant of g would be $ad - bc = a^2 + ab(r + r^q) + b^2r^{1+q} = (a + br)(a + br^q)$ in $\mathbb{F}_{q^2}$ (note that g is in $\mathbb{F}_q$ invertible if and only if it is in $\mathbb{F}_{q^2}$, since the determinant in both fields is the same element in $\mathbb{F}_q$). Since $(a, b) \neq (0, 0)$ and $r, r^q \notin \mathbb{F}_q$, we see that both factors are unequal to 0. Thus $|v_r^G| = \frac{|G|}{|C_G(v_r)|} = \frac{(q^2-1)(q^2-q)}{q^2-1} = q^2 - q$.

All in all we have the following number of distinct elements in the above conjugacy classes

$$(q-1) \cdot 1 + (q-1) \cdot (q^2-1) + \frac{(q-1)(q-2)}{2} \cdot q(q+1) + \frac{q^2-q}{2} \cdot (q^2-q),$$

which equals exactly the group order of $\mathrm{GL}_2(\mathbb{F}_q)$. So we found all conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$. Listed in a table they look like this:

Class rep. g	sI	u_s	$d_{s,t}$	v_r
Size of class	1	q^2-1	$q(q+1)$	q^2-q
No. of classes	$q-1$	$q-1$	$\frac{(q-1)(q-2)}{2}$	$\frac{q^2-q}{2}$

Table 2: Conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$

3.2 Classification of irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ in char 0

In this section we will construct all irreducible characters of $\mathrm{GL}_2(\mathbb{F}_q)$. In the next section we will give two examples of the character table for $q = 2, 3$.

Throughout this section, we will use the following well-known fact:
Let L/K be a field extension. Then two matrices $A, B \in GL_n(K)$ are similar in $GL_n(K)$ if and only if they are similar in $GL_n(L)$. The proof uses the theory of the *Frobenius normal form* of matrices.

Theorem 3.3. *Label the conjugacy classes of* $\mathrm{GL}_2(\mathbb{F}_q)$ *as above and let* $\bar{\cdot} : \mathbb{F}_{q^2}^\times \to K^\times$ *be the map we constructed in the last section. Then the irreducible characters of* $\mathrm{GL}_2(\mathbb{F}_q)$ *are given by* $\lambda_i, \psi_i, \psi_{i,j}, \chi_i$ *as follows:*

	sI	u_s	$d_{s,t}$	v_r
λ_i	$\bar{s}^{2i}$	$\bar{s}^{2i}$	$(\bar{st})^i$	$\bar{r}^{i(1+q)}$
ψ_i	$q\bar{s}^{2i}$	0	$(\bar{st})^i$	$-\bar{r}^{i(1+q)}$
$\psi_{i,j}$	$(q+1)\bar{s}^{i+j}$	$\bar{s}^{i+j}$	$\bar{s}^i\bar{t}^j + \bar{s}^j\bar{t}^i$	0
χ_i	$(q-1)\bar{s}^i$	$-\bar{s}^i$	0	$-(\bar{r}^i + \bar{r}^{iq})$

Table 3: Character table of $\mathrm{GL}_2(\mathbb{F}_q)$

Here we have the following restrictions of the subscripts:
(a) For λ_i *we have* $0 \le i \le q-2$. *Thus there are* $q-1$ *characters of degree* 1.
(b) For ψ_i *we have* $0 \le i \le q-2$. *Thus there are* $q-1$ *characters of degree* q.
(c) For $\psi_{i,j}$ *we have* $0 \le i < j \le q-2$. *Thus there are* $\frac{(q-1)(q-2)}{2}$ *characters of degree* $q+1$.
(d) For χ_i *consider the set* $J := \{1 \le j \le q^2-2 \mid (q+1) \nmid j\}$. *Then for every* $j \in J$, *there exists exactly one* $j' \in J$, *s.t.* $j \equiv q \cdot j' \pmod{q^2-1}$. *This* j' *is unequal to* j *and the unique element* $(j')'$ *corresponding to* j' *is* j *itself. So* J *is divided into* $\frac{|J|}{2} = \frac{q^2-q}{2}$ *disjoint sets of pairs. Take one element of each pair for indexing* χ_i *(it is easy to see that the elements of the same pair yields the same character; use* $\bar{r}^i = \overline{r^i}$*). Thus there are* $\frac{q^2-q}{2}$ *characters of degree* $q-1$.

We will prove the above classification step by step.
The idea is to construct irreducible characters as much as possible and see at the end that we already found all (by summing up the number and seeing it coincides with the number of conjugacy classes).
From now on set $G = \mathrm{GL}_2(\mathbb{F}_q)$.

Lemma 3.4. *There are (at least) q-1 irreducible characters of degree 1 of G.*

Proof. Define $\lambda_i := \sigma_i \circ \det : G \to K^\times$, where σ_i was the irreducible representation of $\mathbb{F}_q^\times$. For $0 \le i \le q-2$ these are all different (just evaluate it at $\left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & 1 \end{smallmatrix}\right)$). $\square$

The values of these characters are exactly those in theorem 3.3.
We will now construct one of the other characters in theorem 3.3, but will show later that they are irreducible and different.

Lemma 3.5. *For all integers i, j there is a character $\psi_{i,j}$ of G with the values described in theorem 3.3.*

Proof. Let $B \leq G$ be the subgroup of upper triangular matrices.[6] We have obviously $|B| = (q-1)^2 q$. Define $\lambda_{i,j} : B \to K^\times$ by

$$\lambda_{i,j} : \begin{pmatrix} s & r \\ 0 & t \end{pmatrix} \mapsto \bar{s}^i \bar{t}^j.$$

The $\lambda_{i,j}$ is an irreducible character of B. We let $\psi_{i,j} := \operatorname{Ind}_B^G \lambda_{i,j}$, which is again a character by corollary 2.24. We will use the definition of induced characters to calculate the values of $\psi_{i,j}$ on each conjugacy class.
We have

$$\psi_{i,j}(sI) = \frac{1}{|B|} \sum_{x \in G} \dot{\lambda}_{i,j}(x^{-1}(sI)x) = \frac{1}{|B|} \sum_{x \in G} \dot{\lambda}_{i,j}(sI) = \frac{|G|}{|B|} \bar{s}^{i+j} = (q+1) \cdot \bar{s}^{i+j}.$$

To compute $\psi_{i,j}(u_s) = \dfrac{1}{|B|} \sum_{x \in G} \dot{\lambda}_{i,j}(x^{-1} u_s x)$ just note that $x^{-1} u_s x \in B$ if and only if $x \in B$, hence

$$\psi_{i,j}(u_s) = \frac{1}{|B|} \sum_{x \in B} \lambda_{i,j}(x^{-1} u_s x) = \frac{1}{|B|} \sum_{x \in B} \lambda_{i,j}(u_s) = \lambda_{i,j}(u_s) = \bar{s}^{i+j}.$$

For $\psi_{i,j}(d_{s,t}) = \dfrac{1}{|B|} \sum_{x \in G} \dot{\lambda}_{i,j}(x^{-1} d_{s,t} x)$ notice that $x^{-1} d_{s,t} x \in B$ if and only if $x = \left(\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right)$ or $x = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$. Denote the set of elements of the first form by D_1 and of the second form by D_2. In the first case (i.e. $x \in D_1$) we get $x^{-1} d_{s,t} x = \left(\begin{smallmatrix} t & * \\ 0 & s \end{smallmatrix}\right)$; in the second case $x^{-1} d_{s,t} x = \left(\begin{smallmatrix} s & * \\ 0 & t \end{smallmatrix}\right)$. Moreover $D_1 \cap D_2 = \emptyset$. This yields to

$$\begin{aligned} \psi_{i,j}(d_{s,t}) = \frac{1}{|B|} \sum_{x \in G} \dot{\lambda}_{i,j}(x^{-1} d_{s,t} x) &= \frac{1}{|B|} \Big(\sum_{x \in D_1} \lambda_{i,j}(x^{-1} d_{s,t} x) + \sum_{x \in D_2} \lambda_{i,j}(x^{-1} d_{s,t} x) \Big) \\ &= \frac{1}{|B|} \Big(\sum_{x \in D_1} \bar{t}^i \bar{s}^j + \sum_{x \in D_2} \bar{s}^i \bar{t}^j \Big) \\ &= \bar{t}^i \bar{s}^j + \bar{s}^i \bar{t}^j. \end{aligned}$$

In the last case we have $\psi_{i,j}(v_r) = \dfrac{1}{|B|} \sum_{x \in G} \dot{\lambda}_{i,j}(x^{-1} v_r x) = 0$, which is obvious because the elements of B do have eigenvalues, but the conjugates of v_r do not have eigenvalues in $\mathbb{F}_q$. $\square$

[6] This group is called the *Borel subgroup* and appears in the *Bruhat decomposition.*

Lemma 3.6. *For each integer i there is an irreducible character ψ_i of G whose values are given in theorem 3.3. For $0 \leq i \leq q-2$ all ψ_i are different.*

Proof. First we need to compute two inner products.
By definition we have

$$\langle \psi_{i,i}, \psi_{i,i} \rangle_G = \frac{1}{|G|} \sum_{x \in G} \psi_{i,i}(x) \psi_{i,i}(x^{-1}).$$

We can split the sum into its conjugacy classes. The first three families of conjugacy classes are enough, since $\psi_{i,i}$ is always zero on the last one as we saw.
For the first family we get the value

$$\psi_{i,i}(sI)\psi_{i,i}((sI)^{-1}) = (q+1)\bar{s}^{2i} \cdot (q+1)\bar{s}^{-2i} = (q+1)^2,$$

for each $s \in \mathbb{F}_q^{\times}$. So the first summand is $(q+1)^2$ multiplied with the number of elements in the first family, so we get $(q+1)^2 \cdot (q-1)$.
For the second family we get the value

$$\psi_{i,i}(u_s)\psi_{i,i}(u_s^{-1}) = \psi_{i,i}(u_s)\psi_{i,i}(u_{s^{-1}}) = \bar{s}^{2i} \cdot \bar{s}^{-2i} = 1,$$

where the first equality holds because $u_s^{-1} \sim u_{s^{-1}}$. So the second summand is the above value multiplied with the number of elements in the second family: $(q-1) \cdot (q^2-1)$.
For the third family we get the value

$$\psi_{i,i}(d_{s,t})\psi_{i,i}(d_{s,t}^{-1}) = 2(\bar{st})^i \cdot 2(\bar{st})^{-i} = 4,$$

The last summand is therefore $4 \cdot \frac{(q-1)(q-2)}{2} \cdot q(q+1)$.
This yields to

$$\langle \psi_{i,i}, \psi_{i,i} \rangle_G = \frac{1}{|G|}((q+1)^2 \cdot (q-1) + (q-1) \cdot (q^2-1) + 4 \cdot \frac{(q-1)(q-2)}{2} \cdot q(q+1)) = 2.$$

In a similar fashion we will compute the inner product

$$\langle \psi_{i,i}, \lambda_i \rangle_G = \frac{1}{|G|} \sum_{x \in G} \psi_{i,i}(x) \lambda_i(x^{-1}).$$

Again, computing the summand for the first family of conjugacy classes yields to

$$\psi_{i,i}(sI)\lambda_i((sI)^{-1}) = (q+1)\bar{s}^{2i} \cdot \bar{s}^{-2i} = (q+1).$$

We have to multiply this value again with the number of elements in the first family of conjugacy classes: $(q+1) \cdot (q-1)$.
For the second family we get the value

$$\psi_{i,i}(u_s)\lambda_i(u_s^{-1}) = \bar{s}^{2i} \cdot \bar{s}^{-2i} = 1.$$

So the second summand is the above value multiplied with the number of elements in the second family: $(q-1)\cdot(q^2-1)$.
For the third family we get the value

$$\psi_{i,i}(d_{s,t})\lambda_i(d_{s,t}^{-1}) = 2(\bar{s}\bar{t})^i \cdot (\bar{s}\bar{t})^{-i} = 2,$$

The last summand is therefore $2 \cdot \frac{(q-1)(q-2)}{2} \cdot q(q+1)$.
This yields to

$$\langle \psi_{i,i}, \lambda_i \rangle_G = \frac{1}{|G|}((q+1)\cdot(q-1) + (q-1)\cdot(q^2-1) + 2\cdot\frac{(q-1)(q-2)}{2}\cdot q(q+1)) = 1.$$

We know that $\psi_{i,i}$ is a character of G, so it is a linear combination of irreducible characters with non-negative integer coefficients. Because of $\langle \psi_{i,i}, \psi_{i,i}\rangle_G = 2$ our character $\psi_{i,i}$ is the sum of exactly two irreducible characters of G with coefficients equal to 1. Since $\langle \psi_{i,i}, \lambda_i\rangle_G = 1$ the irreducible character λ_i is one of the summands. Let ψ_i be the other one, i.e. $\psi_{i,i} = \lambda_i + \psi_i$. The values of ψ_i are already determined by $\psi_i = \psi_{i,i} - \lambda_i$.
The new irreducible characters ψ_i are all different for $0 \le i \le q-2$ (just evaluate them at $d_{\varepsilon,1}$). □

We are now ready to show that the characters $\psi_{i,j}$ are irreducible and pairwise different for the right subscripts (like in theorem 3.3).

Lemma 3.7. *Suppose* $0 \le i < j \le q-2$*, then:*
(i) *The characters* $\psi_{i,j}$ *are irreducible.*
(ii) *The characters* $\psi_{i,j}$ *are all different.*

Proof. (i) We want to show that the inner product in G equals 1. Like in lemma 3.6 we get three summands $\langle \psi_{i,j}, \psi_{i,j}\rangle_G = A + B + C$ with $A = \frac{(q+1)^2(q-1)}{|G|}$, $B = \frac{(q-1)(q^2-1)}{|G|}$ and

$$C = \frac{q(q+1)}{|G|}\cdot\frac{1}{2}\cdot\sum_{s,t\in\mathbb{F}_q^\times, s\neq t}(\bar{s}^i\bar{t}^j + \bar{s}^j\bar{t}^i)(\bar{s}^{-i}\bar{t}^{-j} + \bar{s}^{-j}\bar{t}^{-i}),$$

where the factor $\frac{1}{2}$ comes from double counting the conjugacy classes ($d_{s,t} \sim d_{t,s}$).
To evaluate C note that $D' := \{d_{s,t} \mid s,t \in \mathbb{F}_q^\times\}$ is an abelian subgroup of G of order $(q-1)^2$ (note that $s = t$ is allowed). Then the map $\delta : D' \to K$ defined by $\delta(d_{s,t}) = \bar{s}^i\bar{t}^j + \bar{s}^j\bar{t}^i$ is the sum of two irreducible different characters of degree 1 of D'. Hence

$$2 = \langle \delta, \delta\rangle_{D'} = \frac{1}{|D'|}(4(q-1) + \sum_{s,t\in\mathbb{F}_q^\times, s\neq t}(\bar{s}^i\bar{t}^j + \bar{s}^j\bar{t}^i)(\bar{s}^{-i}\bar{t}^{-j} + \bar{s}^{-j}\bar{t}^{-i})).$$

Thus we get $C = \frac{q-3}{q-1}$ and $A + B + C = 1$. Therefore $\langle \psi_{i,j}, \psi_{i,j}\rangle_G = 1$.

(ii) Let $0 \le k, l \le q-2$ and $0 \le k', l' \le q-2$. Consider again the irreducible characters $\lambda_{k,l}$ of B defined in lemma 3.5. If $(k,l) \neq (k',l')$ then $\lambda_{k,l} \neq \lambda_{k',l'}$ (evaluate both at

$d_{\varepsilon,1}$ and $d_{1,\varepsilon}$).
Remember that we have $0 \le i < j \le q-2$. Take another pair $0 \le i' < j' \le q-2$, we want to show $\psi_{i,j} \neq \psi_{i',j'}$ if $(i,j) \neq (i',j')$. The above observation shows $\lambda_{i,j}+\lambda_{j,i} \neq \lambda_{i',j'}+\lambda_{j',i'}$, otherwise we would have $\lambda_{i',j'} = \lambda_{i,j}$ or $\lambda_{i',j'} = \lambda_{j,i}$, since the irreducible characters form a basis of the class functions. The first case is not allowed based on our assumption. The second case yields to $i' = j$ and $j' = i$, which is not possible because $i' = j > i = j' > i'$.
So there is a matrix $\left(\begin{smallmatrix} s & b \\ 0 & t \end{smallmatrix}\right) \in B$, s.t. either

$$s \neq t \text{ and } \bar{s}^i\bar{t}^j + \bar{s}^j\bar{t}^i \neq \bar{s}^{i'}\bar{t}^{j'} + \bar{s}^{j'}\bar{t}^{i'} \qquad \text{or} \qquad s = t \text{ and } \bar{s}^{i+j} \neq \bar{s}^{i'+j'}.$$

In both cases we get that $\psi_{i,j}$ differs from $\psi_{i',j'}$ on the third or second family of conjugacy classes. $\square$

We constructed so far the first three families of irreducible characters. We will go on with the construction of the last one and need the following

Lemma 3.8. *Let $H = \langle v_\varepsilon \rangle \le G$ be the subgroup of G generated by v_ε. Then $|H| = q^2 - 1$. The group H contains the $q-1$ scalar matrices sI in G, and of the remaining q^2-q elements of H, precisely two belong to each of the conjugacy classes represented by $v_r \sim v_{r^q}$ with $r \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$*

Proof. The order of $v_\varepsilon \in \mathrm{GL}_2(\mathbb{F}_q)$ equals the order of $v_\varepsilon \in \mathrm{GL}_2(\mathbb{F}_{q^2})$. The matrix v_ε has eigenvalues ε and ε^q in $\mathbb{F}_{q^2}$, so the matrix has order $q^2 - 1$.
First note that $v_\varepsilon^i = v_\varepsilon^{iq} \Leftrightarrow q+1|i \Leftrightarrow \varepsilon^i = \varepsilon^{iq}$. The eigenvalues of v_ε^i and of v_ε^{iq} are ε^i and ε^{iq}.
If $i = (q+1)\cdot k$ with $1 \le k \le q-1$, then v_ε^i has eigenvalues $\varepsilon^i = \varepsilon^{iq} \in \mathbb{F}_q$, hence is conjugated to $\varepsilon^i \cdot I$. Since the conjugacy class has size 1, we have equality.
Assume now $1 \le i \le q^2-1$ with $q+1 \nmid i$. Then both $v_\varepsilon^i \neq v_\varepsilon^{iq}$ with $\varepsilon^i \notin \mathbb{F}_q$ are conjugated to v_{ε^i}. $\square$

Lemma 3.9. *For each integer i there exists a character ϕ_i of G which takes the following values:*

	sI	u_s	$d_{s,t}$	v_r
ϕ_i	$q(q-1)\bar{s}^i$	0	0	$\bar{r}^i + \bar{r}^{iq}$

Proof. Let $H = \langle v_\varepsilon \rangle \le G$ and consider the linear character $\alpha_i : H \to K^\times$, which sends the generator v_ε to $\bar{\varepsilon}^i$ (remember that $\bar{\varepsilon} = \omega$ has order q^2-1). Suppose that $g = v_\varepsilon^j \in H$ is conjugated to v_r in G. Then their eigenvalues coincide, i.e. $\{\varepsilon^j, \varepsilon^{jq}\} = \{r, r^q\}$. Then we have $\alpha_i(g) = \alpha_i(v_\varepsilon^j) = \bar{\varepsilon}^{ij} = \bar{\varepsilon^j}^i$, so it is equal to $\bar{r}^i$ or to $\bar{r}^{iq}$ and we get in both cases $\alpha_i(g) + \alpha_i(g^q) = \bar{r}^i + \bar{r}^{iq}$.
Let

$$\phi_i(g) := \mathrm{Ind}_H^G \alpha_i(g) = \frac{1}{|H|}\sum_{x \in G} \dot{\alpha}_i(x^{-1}gx).$$

To compute this expression we will divide G into its four families of conjugacy classes. With lemma 3.8 we know that ϕ_i is zero on the second and third family.
For $g = sI$ with $s \in \mathbb{F}_q^\times$ we get

$$\phi_i(sI) = \frac{1}{|H|}\sum_{x\in G}\dot{\alpha}_i(x^{-1}(sI)x) = \frac{1}{|H|}\sum_{x\in G}\dot{\alpha}_i(sI) = \frac{|G|}{|H|}\alpha_i(sI) = q(q-1)\bar{s}^i,$$

since there exists some integer j with $v_\varepsilon^j = sI$ (as we showed in lemma 3.8) and $\varepsilon^j = s$ (because they have the same eigenvalues).
Let us compute the value on the last family of conjugacy classes:
Let $r \in \mathbb{F}_{q^2}\backslash\mathbb{F}_q$, then lemma 3.8 says there exists a $g \in H$, s.t. $g \sim v_r$. Moreover lemma 3.8 says $g \neq g^q \sim v_r$. Hence

$$\phi_i(v_r) = \phi_i(g) = \frac{1}{|H|}\sum_{x\in G}\dot{\alpha}_i(x^{-1}gx),$$

Now verify easily that $x^{-1}gx \in H \Leftrightarrow x^{-1}gx = g$ or $x^{-1}gx = g^q$. The first equality has $|C_G(g)| = \frac{|G|}{|g^G|} = \frac{|G|}{|v_r^G|} = q^2-1$ solutions for x. The second equation has the same number of solutions for x, since $g \sim v_r \sim g^q$, so there exists an $h \in G$ with $g^q = hgh^{-1}$ and the equation $x^{-1}gx = g^q$ is equivalent to $(xh)^{-1}g(xh) = g$, so we have an obvious bijection between the solutions of both equations. Thus

$$\phi_i(v_r) = \phi_i(g) = \frac{1}{|H|}\sum_{x\in G}\dot{\alpha}_i(x^{-1}gx) = \frac{1}{|H|}((q^2-1)\alpha_i(g) + (q^2-1)\alpha_i(g^q)) = \bar{r}^i + \bar{r}^{iq},$$

as we saw in the beginning of the proof. ∎

Lemma 3.10. *Assume that i is an integer and $(q+1)\nmid i$. Then*

$$\sum_{r\in\mathbb{F}_{q^2}\backslash\mathbb{F}_q}(\bar{r}^i + \bar{r}^{iq})(\bar{r}^{-i} + \bar{r}^{-iq}) = 2(q-1)^2$$

Proof. Note that

$$G_1 = \left\{\begin{pmatrix} r & 0\\ 0 & r^q\end{pmatrix} \mid r \in \mathbb{F}_{q^2}^\times\right\} \text{ and } G_2 = \left\{\begin{pmatrix} r & 0\\ 0 & r^q\end{pmatrix} \mid r \in \mathbb{F}_q^\times\right\}$$

are two abelian groups of order q^2-1 respectively $q-1$. Consider the following sum of two irreducible characters of degree one for each group

$$\chi : \begin{pmatrix} r & 0\\ 0 & r^q\end{pmatrix} \mapsto \bar{r}^i + \bar{r}^{iq}$$

This χ is for G_1 a sum of two different irreducible characters (since $\bar{\varepsilon}^i \neq \bar{\varepsilon}^{iq}$, otherwise $(q+1)\mid i$), hence

$$2 = \langle\chi,\chi\rangle_{G_1} = \frac{1}{q^2-1}\sum_{r\in\mathbb{F}_{q^2}^\times}(\bar{r}^i + \bar{r}^{iq})(\bar{r}^{-i} + \bar{r}^{-iq}).$$

But χ is for the group G_2 twice an irreducible character because $r^q = r$. So

$$4 = \langle \chi, \chi \rangle_{G_2} = \frac{1}{q-1} \sum_{r \in \mathbb{F}_q^\times} (\bar{r}^i + \bar{r}^{iq})(\bar{r}^{-i} + \bar{r}^{-iq}).$$

Putting these formulas together leads to

$$\sum_{r \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q} (\bar{r}^i + \bar{r}^{iq})(\bar{r}^{-i} + \bar{r}^{-iq}) = 2(q^2-1) - 4(q-1) = 2(q-1)^2.$$

$\square$

We are now ready to get the last irreducible characters.

Lemma 3.11. *For each integer i, let χ_i be the class function on G with values like in theorem 3.3. If $(q+1) \nmid i$, then χ_i is an irreducible character of G.*

Proof. Recall the characters $\psi_{i,j}, \psi_i$ and ϕ_i given in lemmas 3.5, 3.6 and 3.9. Define the class function $\chi_i \coloneqq \psi_{0,-i} \cdot \psi_i - \psi_{0,i} - \phi_i$ (where we mean the pointwise multiplication and not the convolution). Just plugin the conjugacy classes to verify that the values are exactly those in *theorem* 3.3.
Next assume $(q+1) \nmid i$ and use the previous lemma with the facts $u_s^{-1} \sim u_{s^{-1}}$ and $v_r^{-1} \sim v_{r^{-1}}$ to compute

$$\langle \chi_i, \chi_i \rangle_G = \frac{1}{|G|}((q-1)^2(q-1) + (q-1)(q^2-1) + (q-1)^2(q^2-q)) = 1.$$

Remember that the pointwise product of two characters is again a character (the character of the tensor product), hence χ_i is a linear combination of characters with integer coefficients, splitting these into the irreducible characters leads to a linear combination of irreducible characters with integer coefficients. But we know $\langle \chi_i, \chi_i \rangle_G = 1$, hence χ_i is the linear combination of exactly one irreducible character with coefficient either 1 or -1. But we also know that $\chi_i(1) = q - 1 > 0$ and that a character evaluated at 1 gives the dimension of the representation back, hence only 1 as a coefficient is possible. That means χ_i is an irreducible character. $\square$

Let us now show that the new constructed irreducible characters are different for the subscripts described in theorem 3.3.

Lemma 3.12. *Suppose that i and j are integers from the disjoint sets that we described in theorem 3.3, i.e. $1 \le i \ne j, \le q^2 - 2$, with $(q+1) \nmid i$, $(q+1) \nmid j$ and $i \not\equiv jq \pmod{q^2-1}$. Then the characters χ_i and χ_j of G are different.*

Proof. Consider again the irreducible linear characters α_k of $H = \langle v_\varepsilon \rangle \le G$ described in lemma 3.9. We have $\alpha_i \ne \alpha_j$ and $\alpha_i \ne \alpha_{jq}$ (evaluate both at v_ε). Hence $\alpha_i + \alpha_{iq} \ne \alpha_j + \alpha_{jq}$ (because they form a basis in the space of class functions on H). So there is a matrix $g \in H$, s.t. either

$g = sI$ with $s \in \mathbb{F}_q^\times$ and $(\alpha_i + \alpha_{iq})(g) = 2\bar{s}^i \neq 2\bar{s}^j = (\alpha_j + \alpha_{jq})(g)$
or
$g \sim v_r$ with $r \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ and $(\alpha_i + \alpha_{iq})(g) = \bar{r}^i + \bar{r}^{iq} \neq \bar{r}^j + \bar{r}^{jq} = (\alpha_j + \alpha_{jq})(g)$.
In both cases we get that χ_i differs from χ_j on the first or last family of conjugacy classes. $\square$

We have now completed the proof of theorem 3.3, since we have shown that the class functions given in the theorem are different irreducible characters; and the number of them equals the number of the conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$.

Although the character table of $\mathrm{GL}_2(\mathbb{F}_q)$ was first given 1907, it was not until the 1950's that the character table of $\mathrm{GL}_3(\mathbb{F}_q)$ was found. Then, in 1955, James Alexander Green determined the character table of $\mathrm{GL}_n(\mathbb{F}_q)$ over the complex numbers for all positive integers n in his paper [Gre55].

3.3 Example of character table for $q = 2, 3$

Since the description of the character table was in some sense abstract, let us take a look at the concrete cases $q = 2$ and $q = 3$.

In the first case let $\mathbb{F}_4$ be realized through $\mathbb{F}_2[X]/(X^2 + X + 1)$, since the polynomial $X^2 + X + 1$ is irreducible in $\mathbb{F}_2[X]$. Instead of writing $\overline{X}$ for the residue class of X in $\mathbb{F}_4$, we will just write X to get not confused with the map $\bar{\cdot} : \mathbb{F}_{q^2}^\times \to K^\times$ defined in the beginning of section 3.
It is easy to see $\mathbb{F}_4^\times = \langle X \rangle$. Let $\varepsilon = X$ be the generator of $\mathbb{F}_4^\times$ and ω be a primitive third root in $K^\times$ (i.e. $\mathrm{ord}(\omega) = 3$).
Regarding the classification in theorem 3.3, we get a 3×3 matrix:

	I	u_1	v_X
λ_0	1	1	1
ψ_0	2	0	-1
χ_1	1	-1	1

Table 4: Character table of $\mathrm{GL}_2(\mathbb{F}_2)$

The computation is straightforward. Just use the facts $\bar{1} = 1$ and $\omega^2 + \omega + 1 = 0$. This character table might be familiar. Indeed, we have the same character table for S_3 showed in table 1. The reason is that $\mathrm{GL}_2(\mathbb{F}_2)$ and S_3 are isomorphic as groups, since $\mathrm{GL}_2(\mathbb{F}_2)$ operates faithful on the set $V = \{\binom{1}{0}, \binom{0}{1}, \binom{1}{1}\}$, i.e. there is an injective group homomorphism $\mathrm{GL}_2(\mathbb{F}_2) \to S(V)$. The groups are of the same cardinality, so they are isomorphic.
Note that the inverse of the above observation with the character table does not hold, that means there are non-isomorphic groups with the same character table. An example is the dihedral group of order $2n = 8$ and the quaternion group given in [Ste11, p.101-103].

Let us go on with the case $q = 3$. this table is much bigger; it is of size 8×8.
First consider the finite field $\mathbb{F}_9 = \mathbb{F}_3[X]/(X^2+1)$. Let us again write X instead of $\overline{X}$. Then $\mathbb{F}_9^\times = \langle X+1 \rangle$, because:

$(X+1)^1 = X+1$	$(X+1)^2 = 2X$	$(X+1)^3 = 2X+1$	$(X+1)^4 = 2$
$(X+1)^5 = 2X+2$	$(X+1)^6 = X$	$(X+1)^7 = X+2$	$(X+1)^8 = 1$

Define now $r_1 := X+1$, $r_2 := 2X$, $r_5 := X+2$ to be the representatives of the fourth family of conjugacy classes. The computation of the character table is again very easy, just use the facts $\bar{2} = \omega^4 = -1$, $\overline{v_{r_1}} = \omega$, $\overline{v_{r_2}} = \omega^2$ and $\overline{v_{r_5}} = \omega^5$. We get the following character table:

	I	$2I$	u_1	u_2	$d_{1,2}$	v_{r_1}	v_{r_2}	v_{r_5}
λ_0	1	1	1	1	1	1	1	1
λ_1	1	1	1	1	-1	-1	1	-1
ψ_0	3	3	0	0	1	-1	-1	-1
ψ_1	3	3	0	0	-1	1	-1	1
$\psi_{0,1}$	4	-4	1	-1	0	0	0	0
χ_1	2	-2	-1	1	0	$-(\omega+\omega^3)$	0	$\omega+\omega^3$
χ_2	2	2	-1	-1	0	0	2	0
χ_5	2	-2	-1	1	0	$\omega+\omega^3$	0	$-(\omega+\omega^3)$

Table 5: Character table of $\mathrm{GL}_2(\mathbb{F}_3)$

And we have that $z := \omega + \omega^3$ satisfies $z^2 = -2$.

4 Modular representations of $\mathrm{GL}_2(\mathbb{F}_q)$

Now we arrived at the last section, where we discuss the difference of our theory in characteristic 0 and in characteristic p', where p' divides the group order.
As before consider K to be algebraically closed, G to be finite. Then we have the following

Proposition 4.1 ([Alp93, p. 14])**.** *The number of irreducible representations of G equals the number of conjugacy classes of G of elements or order not divisible by the characteristic of K.*

Let us investigate the number of p-regular conjugacy classes (that of order coprime to p) of $G = \mathrm{GL}_2(\mathbb{F}_q)$ (with p divides q).
We will go through all four families of conjugacy classes beginning with the first:
It is easy to see that $\mathrm{ord}(sI) = \mathrm{ord}(s) \mid q-1$, hence coprime to p.
For the second family, note that $u_s^k = \left(\begin{smallmatrix} s^k & ks^{k-1} \\ 0 & s^k \end{smallmatrix}\right)$, hence $\mathrm{ord}(u_s) = \mathrm{lcm}(p, \mathrm{ord}(s)) = p \cdot \mathrm{ord}(s)$. So divisible by p.
For the third one we have $\mathrm{ord}(d_{s,t}) = \mathrm{lcm}(\mathrm{ord}(s), \mathrm{ord}(t)) \mid q-1$, hence again coprime to p.
For the last family we get $\mathrm{v_r} = \mathrm{ord}(r) \mid q^2-1$, thus coprime to p.
All in all we have

$$(q-1) + \frac{(q-1)(q-2)}{2} + \frac{q^2-q}{2} = q^2-q$$

p-regular conjugacy classes.
Note that our developed theory is not applicable to determine if a representation is irreducible or not. Or if two irreducible representations are isomorphic or not. So new methods would be required to do this. Since this thesis give no introduction into modular representation theory, we have to compute representations by hand without any machinery. As we see above, the growth of the p-regular classes is quadratic, so let us only consider the case $q = 2$.
We saw in table 1 that $\mathrm{GL}_2(\mathbb{F}_2) \cong S_3$ has three irreducible different characters if $\mathrm{char}(K) = 0$. But if we consider the case $\mathrm{char}(K) = 2$, then χ_1 and χ_2 coincide. The proof that χ_3 is irreducible we gave in example 2.20, also works for all fields K with $\mathrm{char}(K) \neq 3$ (because we divided by 3).
With the proposition above, we found all irreducible characters of $\mathrm{GL}_2(\mathbb{F}_2) \cong S_3$ in the modular case $\mathrm{char}(K) = 2$.
Consider again $\mathrm{GL}_2(\mathbb{F}_2) \cong S_3$, but now in characteristic 3, then there are two 3-regular conjugacy classes. Hence χ_1 and χ_2 are all irreducible characters.
Now we have a phenomenon in the modular case, that we do not have in the regular case: Maschke's theorem is not true anymore.

Theorem 4.2. *Let U be the subrepresentation of $\mathrm{GL}_2(\mathbb{F}_2) \cong S_3$ given in example 2.20 and let $\mathrm{char}(K) = 3$. Then this representation is reducible, but not decomposable.*

Proof. Let $W = \langle v_1 + v_2 + v_3 \rangle$ be a subrepresentation of U. It is easy to show that if there is another one dimensional subrepresentation of U, then it has to be W. Hence U is not irreducible, but indecomposable. $\square$

Let us go back to the case $\mathrm{char}(K) = 2$ and show that Maschke's theorem fails here as well.
Just consider the projection $S_3 \twoheadrightarrow S_3/\langle(123)\rangle \cong (\mathbb{F}_2, +)$ and compose it with the representation given in example 1.23.

All in all we see that the modular representations behave different than that of the regular representations. But they are not unrelated, since there are methods to lift representations of characteristic p to characteristic zero and vice versa reducing representations of characteristic zero to characteristic p. This is the main idea of the so called *Brauer Theory* described in [Ser77].

References

[Alp93] Jonathan L. Alperin. *Local representation theory: Modular representations as an introduction to the local representation theory of finite groups*. Cambridge University Press, 1993.

[Bos13] Siegfried Bosch. *Algebra*. SpringerSpektrum, 2013.

[Gre55] James Alexander Green. “The Characters of the Finite General Linear Groups”. In: *American Mathematical Society* (1955), pp. 402–447 (46 pages). URL: https://www.jstor.org/stable/1992997.

[JL01] Gordon James and Martin Liebeck. *Representations and Characters of Groups*. Cambridge University Press, 2001.

[Ser77] Jean-Pierre Serre. *Linear representations of finite groups*. Vol. 42. Springer, 1977.

[Ste11] Benjamin Steinberg. *Representation Theory of Finite Groups: An Introductory Approach*. Springer Science & Business Media, 2011.