Quantum Walks

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Motivation

Mathematician George Pólya liked to take morning walks through the woods. He noticed that he would regularly bump into the same couple.
Then he asked himself "what is the probability that these two randomly walking groups bump into each other?"
We need a mathematical framework to answer such questions.

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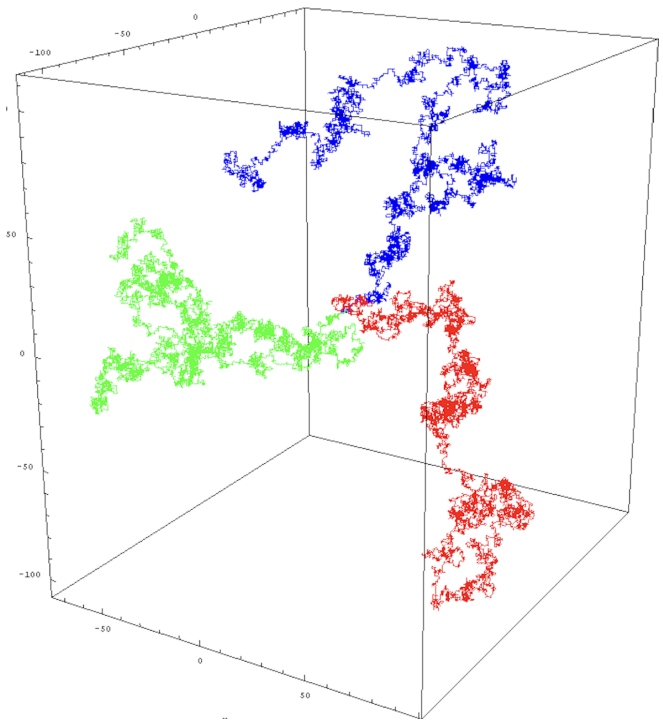

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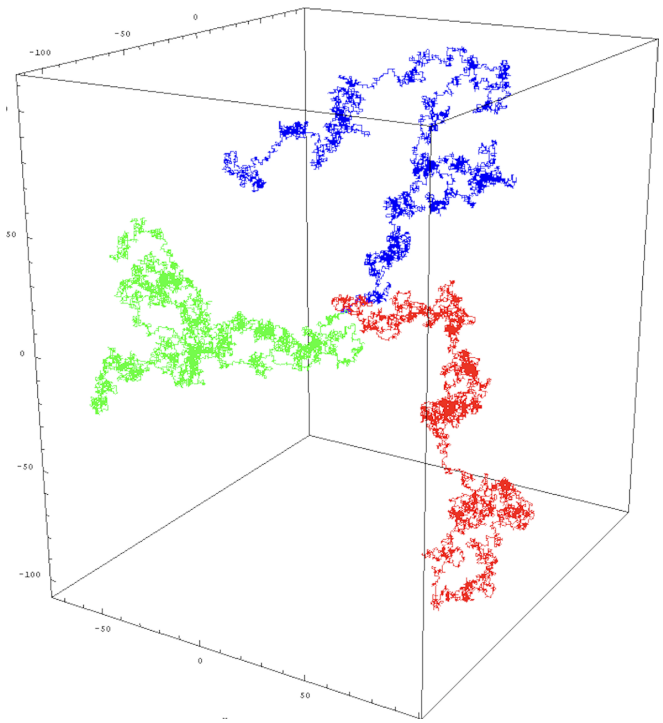

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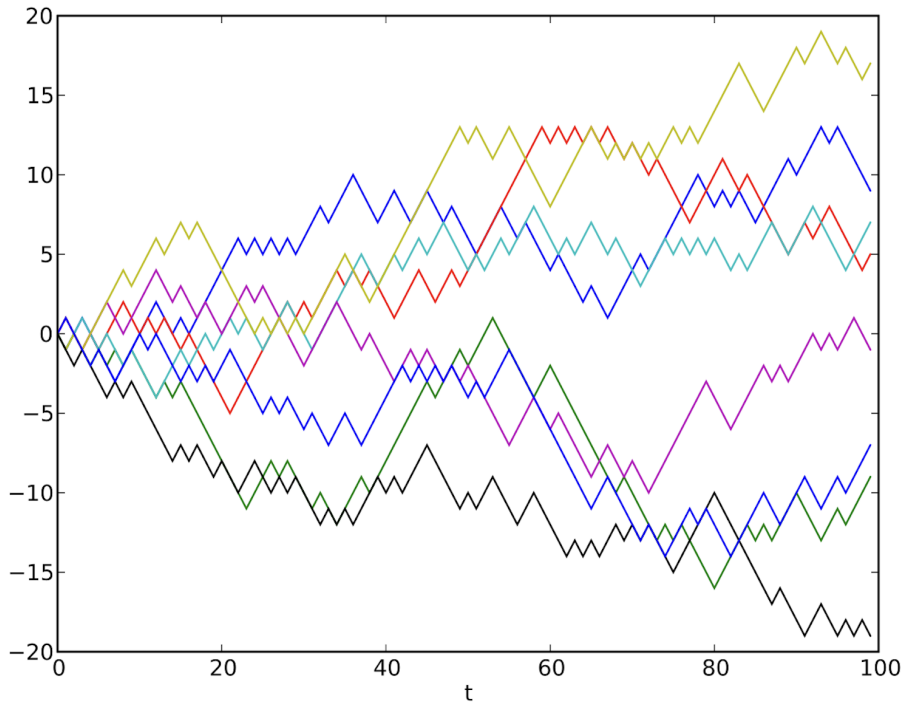

[https://de.wikipedia.org/wiki/Random_Walk]
Black–Scholes model: stock market prices evolve according to a random walk.

Motivation

Shuffling a deck of cards can be modeled by a random walk.
Fun fact: 7 *riffle shuffles* are sufficient to consider the deck as "good shuffled".
For *overhand shuffle* we need 10000.

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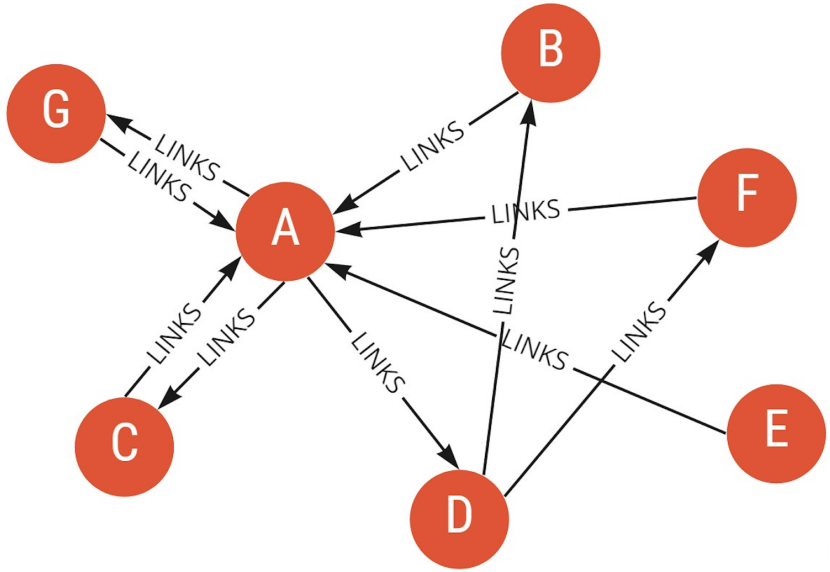
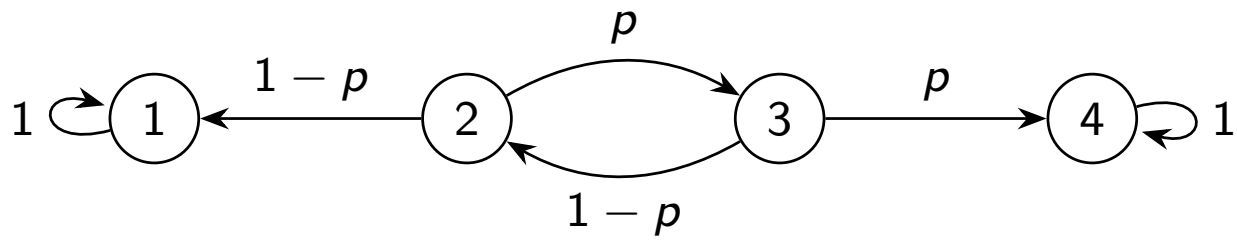
Google's search algorithm *PageRank*: measure the importance of a webpage using random surfer model.



[https://www.pulsem.fr/pagerank-toujours-important-seo/]

Classical Random Walks

- **Informally**: a random walk is a series of steps through a state space, where the location of the next step is chosen randomly from among a set of available options.



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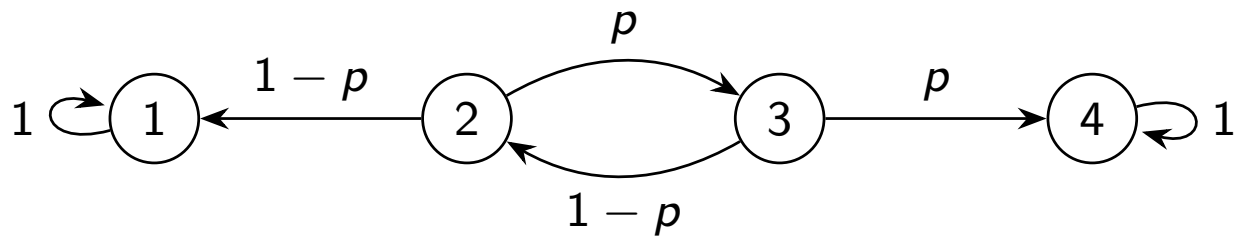


- We can define random walks on very different state spaces:
  d dimensional lattice $\mathbb{Z}^d$, graphs, mathematical groups, Riemannian manifolds,...
- We can have discrete-time or continuous-time (even more exotic index sets).
- We can add many features (e.g. absorbing states, reflecting states,...).

Classical Random Walks on $\mathbb{Z}^d$

1-dimensional simple random walk on $\mathbb{Z}$

Initial position $S_0 \in \mathbb{Z}$.
After each time step: go to the right with probability p or to the left with $1-p$.
Let S_n be the position after n steps.
Model it mathematically by:

$$X_i \overset{\text{iid}}{\sim} \text{Bernoulli}(p)$$

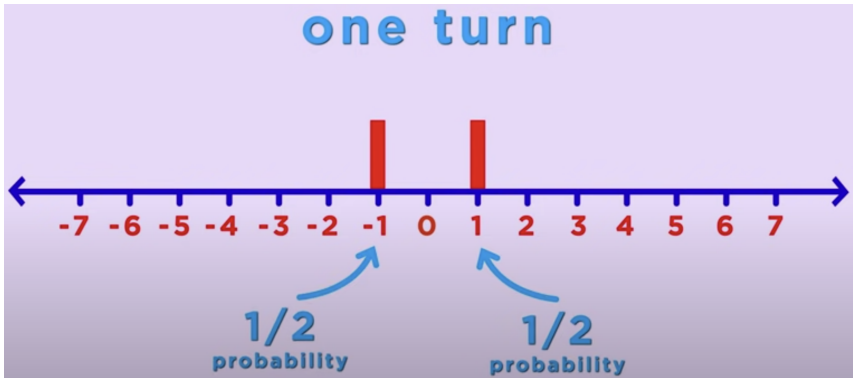
with $\Pr[X_i = 1] = p$ and $\Pr[X_i = -1] = 1 - p$.
Hence our position after n steps is

$$\begin{aligned} S_n &= S_{n-1} + X_n \\ &= S_0 + X_1 + X_2 + \ldots + X_n. \end{aligned}$$

- **Example**:
  How does S_n evolve over time for $S_0 = 0$ and $p = 1/2$?

Classical Random Walks on $\mathbb{Z}^d$

[YouTube - PBS Infinite Series - What is a Random Walk?]



S_1 after one time step

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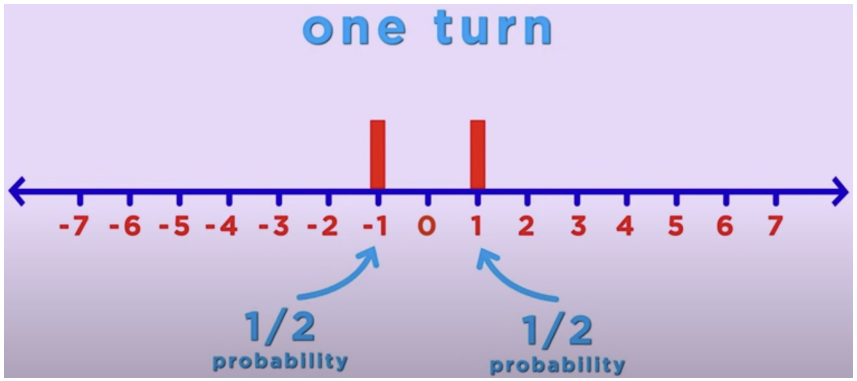

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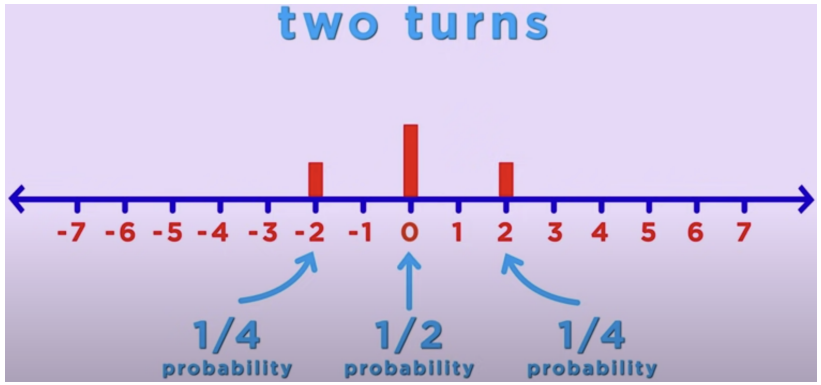

S_2 after two time steps

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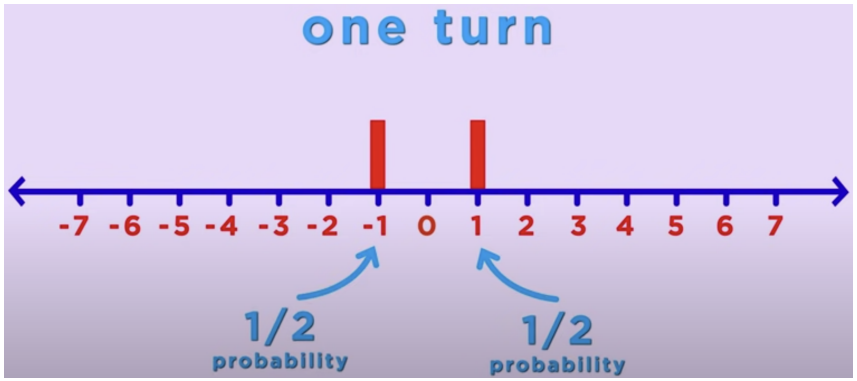

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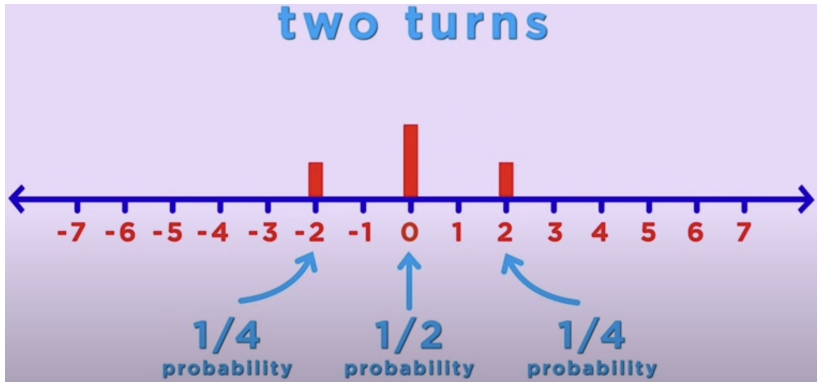
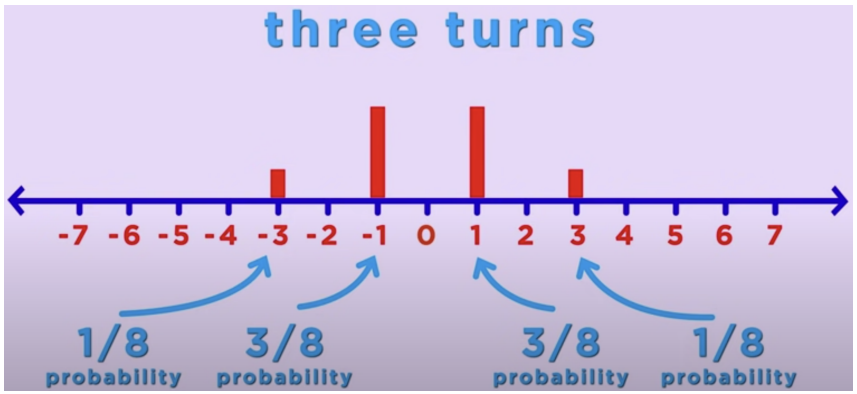


S_2 after two time steps



S_3 after three time steps

Classical Random Walks on $\mathbb{Z}^d$

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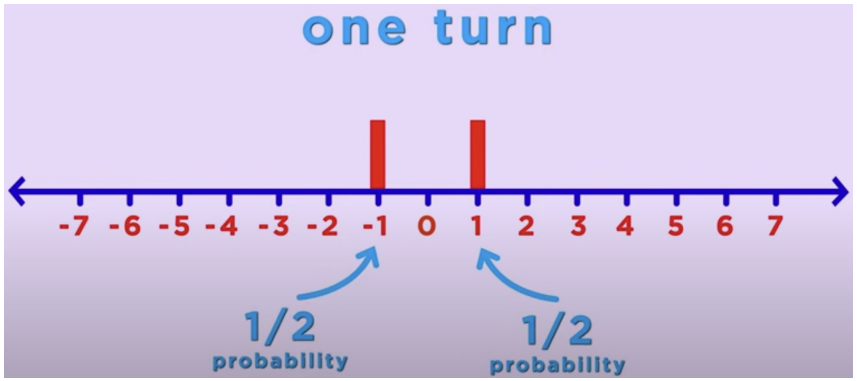

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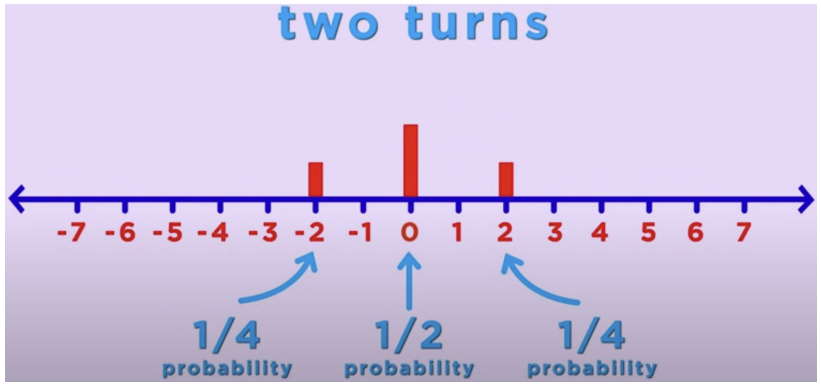

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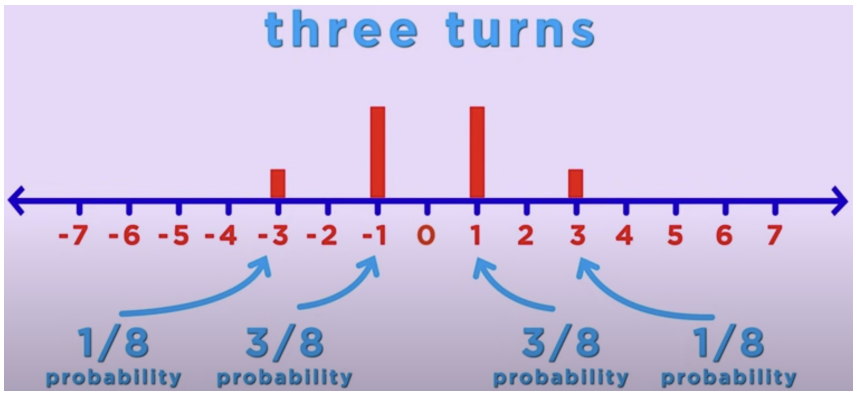

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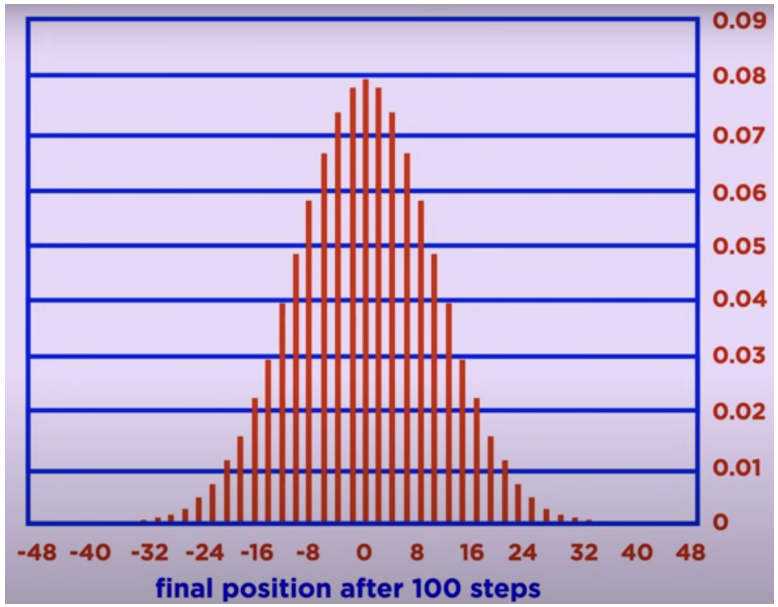

S_{100} after 100 time steps

Classical Random Walks on graphs

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$$\Pr[x, y] = \begin{cases} 1/d, & \text{if } \{x, y\} \in E \\ 0, & \text{else} \end{cases}$$

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- *Initial distribution*: unlike before, starting point X_0 is not determined.
  It is chosen from a probability distribution $X_0 \sim p_0$, i.e. p_0 is a probability vector over the vertices V.
- We are interest in how X_n evolves over time. Let's compute X_1:

$$\Pr[X_1 = y] = \sum_x \Pr[X_0 = x] \cdot \Pr[x, y] = (p_0 \cdot P)[y].$$

  Hence

$$X_1 \sim p_0 \cdot P \Longrightarrow X_n \sim p_0 \cdot P^n.$$

Stationary distribution

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Uniqueness and Convergence

i) Stationary distribution π is unique (here it is used that G is connected).
ii) If G is additionally non-bipartite, then $p_n \xrightarrow{n \to \infty} \pi \ \ \forall p_0$.

Mixing time and hitting time

- The $\epsilon-$*mixing time* measures how fast we approach the stationary distribution with an error of at most ϵ:

$$\mathrm{MT}(\epsilon) := \min\{n : \sup_{p_0} \|p_0 \cdot P^n - \pi\| \leq \epsilon\}.$$

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Convergence rate

$$\mathrm{MT}(\epsilon) \leq \frac{1}{\delta} \cdot (\frac{1}{2} \log(N) + \log\left(\frac{1}{\epsilon}\right))$$

Mixing time and hitting time

- Let $M \subseteq V$ be a subset of the vertices, which are called *marked*.
  Assume we start with the stationary distribution and want to hit an element in M.
  The *hitting time* asks for how fast we can find such a marked element

$$\tau_M := \min_{t \geq 0} \{ t \mid X_t \in M \wedge X_0 \sim \pi \},$$

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- This *random walk search algorithm* can find marked elements in $O\left(\frac{1}{\delta} \cdot \frac{N}{|M|}\right)$.
  Start at vertex x, then repeat until find marked vertex: check if current vertex is marked if not, run a random walk for roughly $1/\delta$ steps to get close to the uniform distribution.
  Can we improve this by switching to a quantum setting?

Quantum Walks: quantizing random walks (part 1)

- How to convert random walk into a quantum walk?
- **Naive approach** would be a unitary U of the form $|x\rangle \mapsto \frac{1}{\sqrt{d}} \sum_{y \sim x} |y\rangle$.

  But this cannot exist in general: consider a graph with two different vertices x and z, which have the same neighbours. So $U|x\rangle = U|z\rangle$, hence $|x\rangle = |z\rangle$. ↯

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- Classical random walk on $\mathbb{Z}$: coin flip followed by a shift.
  Solution: combine them into one object. Add *coin space* to the *state space* and define two unitaries.
  Think of it as: position of the walker & direction that the walker points to.

Quantum Walks: quantizing random walks (part 2)

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- **Coin toss**: set $|\psi_x\rangle = \frac{1}{\sqrt{d}} \sum_{y\sim x} |x, y\rangle$. Define the coin toss by

$$C = 2\left(\sum_{x\in V} |\psi_x\rangle\,\langle\psi_x|\right) - I,$$

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  which is a reflection around $|\psi_x\rangle$.
- Let the quantum walk operator be $W = S \cdot C$.

Quantum walk on $\mathbb{Z}$

- Let $\mathcal{H}_P$ be the Hilbert space spanned by the positions of the walker. On the line it is spanned by $\{|i\rangle : i \in \mathbb{Z}\}$.
  Add the coin space $\mathcal{H}_C$ to it, spanned by $\{|0\rangle, |1\rangle\} = \{|\rightarrow\rangle, |\leftarrow\rangle\}$.
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- As a coin flip, we will take the *Hadamard coin* $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.
- Start with state $|\Psi_0\rangle = |\rightarrow\rangle \otimes |0\rangle$, apply $H \otimes I$ to get $\frac{1}{\sqrt{2}}|\rightarrow\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|\leftarrow\rangle \otimes |0\rangle$.
  Apply S to get $|\Psi_1\rangle = \frac{1}{\sqrt{2}}|\rightarrow\rangle \otimes |1\rangle + \frac{1}{\sqrt{2}}|\leftarrow\rangle \otimes |-1\rangle$.

Quantum walk on $\mathbb{Z}$

- Measure to get $\{|\rightarrow\rangle \otimes |1\rangle, |\leftarrow\rangle \otimes |-1\rangle\}$ each with probability $1/2$.
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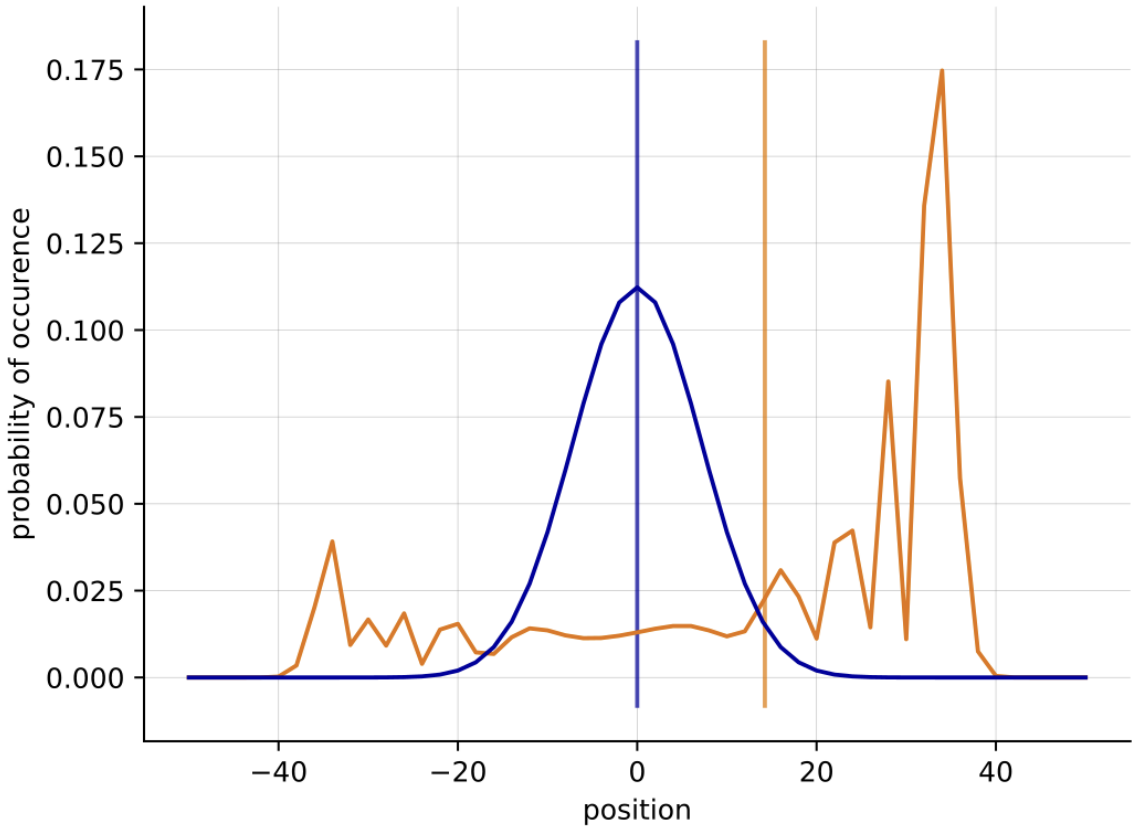
- What does happen if we do not measure in-between?
  It will behave completely different!
- Let $U = S \cdot (H \otimes I)$, then we get

$$\begin{aligned} |\Psi_3\rangle &= U^3 |\Psi_0\rangle \\ &= \frac{1}{2\sqrt{2}} \left(|\rightarrow\rangle \otimes |3\rangle + |\leftarrow\rangle \otimes |1\rangle + 2 |\rightarrow\rangle \otimes |1\rangle - |\rightarrow\rangle \otimes |-1\rangle + |\leftarrow\rangle \otimes |-3\rangle \right). \end{aligned}$$

So $|3\rangle, |-3\rangle, |-1\rangle$ appear with probability $1/8$, but $|1\rangle$ with $5/8$.

Quantum walk on $\mathbb{Z}$

Classical random walk vs. quantum walk after 50 steps:



[https://en.wikipedia.org/wiki/Quantum_walk]

Reason for drifting to the right: the Hadamard coin treats $|\rightarrow\rangle$ and $|\leftarrow\rangle$ differently. $|\leftarrow\rangle$ will appear more often with negative and positive coefficients, which will cancel out.

Quantum walk searching algorithm

- Remember $|\psi_x\rangle = \frac{1}{\sqrt{d}} \sum_{y\sim x} |x,y\rangle$, $C = 2\left(\sum_{x\in V} |\psi_x\rangle \langle\psi_x|\right) - I$ and $W = S \cdot C$.

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- Our *quantum walk algorithm* will work similar to Grover's algorithm. Define two states

$$|G\rangle := \frac{1}{\sqrt{|M|}} \sum_{x\in M} |\psi_x\rangle\,, \quad |B\rangle := \frac{1}{\sqrt{N-|M|}} \sum_{x\notin M} |\psi_x\rangle\,.$$

And

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- Now we only have to implement the reflection through $|B\rangle$ and through $|U\rangle$, like in Grover.
  The first one is easy: just check if the first register has a marked element. If yes, put a -1.
  The second one is harder. Here we will use the quantum walk W.

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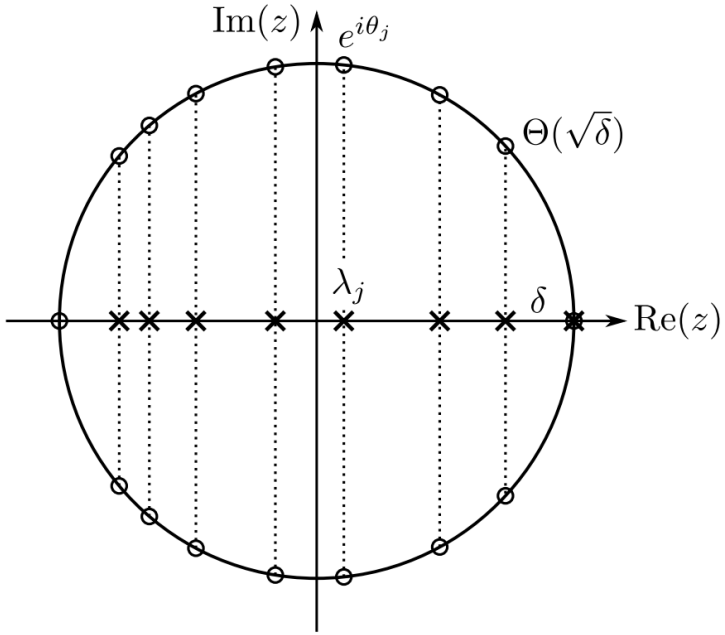
  The second one is harder. Here we will use the quantum walk W.
- We have that $|\pi\rangle = \frac{1}{\sqrt{Nd}} \sum_{(x,y)\in E} |x,y\rangle$ is a stationary state of W.

Quantum walk searching algorithm

Szegedy's spectral lemma (2004)

The non-trivial spectrum of the quantum walk operator W can be characterized as follows: for every eigenvalue $\lambda_j = \cos\theta_j \neq 1$ of P, the quantum walk operator W has a pair of eigenvalues $e^{\pm i\theta_j}$ with $\theta_j \in (0, \pi)$.

We have a correspondence between the eigenvalues of P and of W by projecting them to the circle.
The *phase gap* is defined by
$\Delta := \min\{\theta_j \mid \theta_j \neq 0\}$.
By the lemma we have $\theta_j = \arccos(\lambda_j)$.
Hence $\Delta = \arccos(\lambda_2)$.



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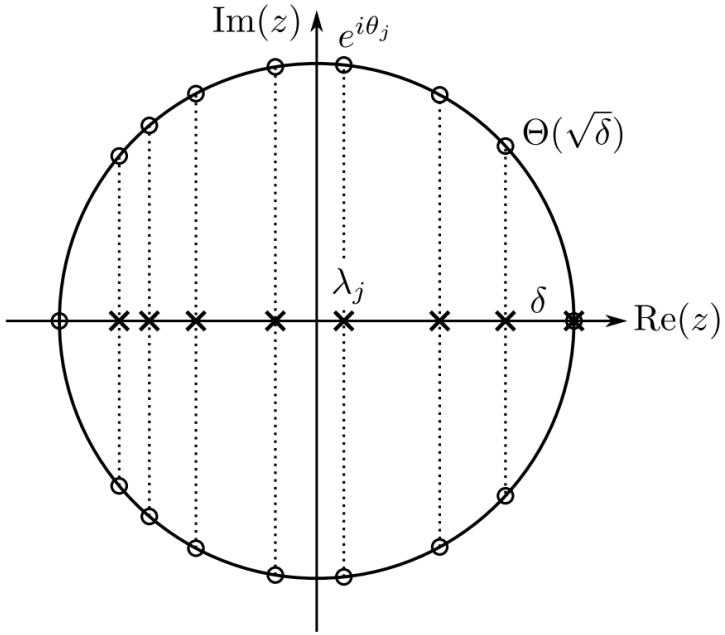

- Now use $\frac{x^2}{2} \geq 1 - \cos(x) \;\; \forall x \in \mathbb{R}$ to get $x \geq \sqrt{2 \cdot (1 - \cos(x))}$. Plug in Δ to get

$$\Delta \geq \sqrt{2 \cdot (1 - \lambda_2)} = \sqrt{2\delta}.$$

Quantum walk searching algorithm

- For the analysis(!) use spectral decomposition for any state $|\Psi\rangle = \alpha_0 |\pi\rangle + \sum \alpha_j |v_j\rangle$ with $W |v_j\rangle = e^{i\theta_j} |v_j\rangle$ and $\theta_j \geq \Delta > 0$.

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- All in all our quantum walk searching algorithm needs

$$O\left(\frac{1}{\Delta} \cdot \sqrt{\frac{N}{|M|}}\right) = O\left(\sqrt{\frac{1}{\delta} \cdot \frac{N}{|M|}}\right)$$

.

Convert problem into graph

- High level idea how to solve problems using quantum walks:
  1) Convert problem P into a graph G with marked elements M, s.t. "Finding marked element in G" $\Leftrightarrow$ "Finding solution of P".
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  "Finding marked element in G" $\Leftrightarrow$ "Finding solution of P".
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- Solve *unstructured search problem* using quantum walks (instead of Grover).
  Given an N-bit string with exactly T 1's. We want to find a 1.
  Convert to N-complete graph with N vertices. It should be complete because we have the freedom to query any index in the list, meaning we can jump to any other position/vertex.
  Now the marked elements corresponds to the 1's, meaning $|M| = T$. Moreover one can show $\delta = 1 - \frac{1}{N-1} \approx 1$. Hence the complexity to find a marked element is

$$O\left(\sqrt{\frac{N}{|M|}}\right) = O\left(\sqrt{\frac{N}{T}}\right).$$

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End

Thank you for your attention :-)
Questions?